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Hidden Borcherds symmetries in $\mathbb{Z}_n$ orbifolds of M-theory and magnetized D-branes in type 0' orientifolds

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Abstract: We study $T^{11-D-q} \times T^q/\mathbb{Z}_n$ orbifold compactifications of eleven-dimensional supergravity and M-theory using a purely algebraic method. Given the description of maximal supergravities reduced on square tori as non-linear coset σ-models, we exploit the mapping between scalar fields of the reduced theory and directions in the tangent space over the coset to construct the orbifold action as a non-Cartan preserving finite order inner automorphism of the complexified U-duality algebra. Focusing on the exceptional serie of Cremmer-Julia groups, we compute the residual U-duality symmetry after orbifold projection and determine the reality properties of their corresponding Lie algebras. We carry out this analysis as far as the hyperbolic $\mathfrak{e}_{10}$ algebra, conjectured to be a symmetry of M-theory. In this case the residual subalgebras are shown to be described by a special class of Borcherds and Kac-Moody algebras, modded out by their centres and derivations. Furthermore, we construct an alternative description of the orbifold action in terms of equivalence classes of shift vectors, and, in $D = 1$, we show that a root of $\mathfrak{e}_{10}$ can always be chosen as the class representative. Then, in the framework of the $E_{10|10}/K(E_{10|10})$ effective σ-model approach to M-theory near a spacelike singularity, we identify these roots with brane configurations stabilizing the corresponding orbifolds. In the particular case of $\mathbb{Z}_2$ orbifolds of M-theory descending to type 0' orientifolds, we argue that these roots can be interpreted as pairs of magnetized D9- and D9'-branes, carrying the lower-dimensional brane charges required for tadpole cancellation. More generally, we provide a classification of all such roots generating $\mathbb{Z}_n$ product orbifolds for $n \leqslant 6$, and hint at their possible interpretation.

Keywords: Solitons Monopoles and Instantons, Intersecting branes models, M-Theory, Global Symmetries.

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1. Introduction

Hidden symmetries in toroidally compactified maximal supergravity theories have been known for a long time, since the foundating works of Cremmer and Julia [1−5]. In particular, the bosonic part of $11D$ supergravity compactified on T^{11-D} for $10 \geqslant D \geqslant 3$ was shown to possess a continuous exceptional $E_{11-D|11-D}(\mathbb{R})$ global symmetry, provided all RR- and NS-NS fields are dualized to scalars whenever possible [6, 7]. This serie of exceptional groups also appear as symmetries of the action of type IIA supergravity compactified on T^{10-D}, and the BPS states of the compactified theory turn out to arrange into multiplets of the Weyl group of $E_{11-D|11-D}(\mathbb{R})$ [8]. From the type IIA point of view, this continuous symmetry does not preserve the weak coupling regime in g_s, and is thus expected to be broken by quantum effects. Nevertheless, the authors of [9] have advocated that a discrete version thereof, namely $E_{11-D|11-D}(\mathbb{Z})$ remains as an exact quantum symmetry of both $11D$ supergravity and type II theories, and thus might provide a guideline for a better understanding of M-theory.

This exceptional serie of arithmetic groups can alternatively be obtained as the closure of the perturbative T-duality symmetry of IIA theory compactified on T^{10-D}, namely $SO(10-D, 10-D, \mathbb{Z})$, with the discrete modular group of the $(11-D)$-torus of M-theory, namely $SL(11-D, \mathbb{Z})$. In $D = 9$, the latter turns into the expected S-duality symmetry of the dual IIB string theory. In this perspective, the global $E_{11-D|11-D}(\mathbb{Z})$ can be regarded as a unifying group encoding both a target-space symmetry, which relates apparently different string backgrounds endowed with isometries, and a rigid symmetry of the maximally symmetric space of compactification which naturally contains a non-perturbative symmetry of type IIB string theory (which is, by the way, also shared by the heterotic string compactified to four dimensions). Furthermore, this so-called U-duality symmetry has been conjectured to extend to the moduli space of M-theory compactified on T^{11-D} for $10 \geqslant D \geqslant 3$. In particular, it was shown in [10] how to retrieve exact R^4 and $R^4 H^{4g-4}$ corrections as well as topological couplings [11], from M-theory $E_{11-D|11-D}(\mathbb{Z})$-invariant mass formulae.

In $D = 2, 1$, the dualization procedure mentioned above is not enough to derive the full U-duality symmetry, which has been conjectured, already some time ago [12 − 14] to be described by the Kac-Moody affine $\mathfrak{e}_{9|10}$ and hyperbolic $\mathfrak{e}_{10|10}$ split forms, that are characterized by an infinite number of positive (*real*) and negative/null (*imaginary*) norm roots. In a more recent perspective, $\mathfrak{e}_{10|10}$ and the split form of $\mathfrak{e}_{11}$ have been more generally put forward as symmetries of the uncompactified $11D$, type II and type I supergravity theories, and possibly as a fundamental symmetry of M-theory itself, containing the whole chain of Cremmer-Julia split algebras, and hinting at the possibility that M-theory might prove intrinsically algebraic in nature.

Along this line, [15] have proposed a tantalizing correspondence between classical $11D$ supergravity operators at a given point close to a spacelike singularity and the coordinates of a one-parameter sigma-model based on the coset $\mathfrak{e}_{10|10}/\mathfrak{k}(\mathfrak{e}_{10|10})$ and describing the dynamics of a hyperbolic cosmological billiard. In particular, a class of real roots of $\mathfrak{e}_{10}$ have been identified, using a BKL expansion [16, 17], as multiple gradients of $11D$ supergravity fields reproducing the truncated equations of motion of the theory. More recently [18], imaginary roots were shown to correspond to 8th order $R^m(DF)^n$ type M-theory corrections to the classical $11D$ supergravity action.

From this perspective, studying the behaviour of the infinite dimensional U-duality symmetry $\mathfrak{e}_{10|10}$ on singular backgrounds is a promising direction. This paper is the first in a serie of two papers aiming at obtaining an algebraic characterization of M-theory compactified on various orbifolds. More precisely, we are interested in studying the residual U-duality symmetry that survives the orbifold projection and the way the shift vectors defining the orbifold action are related to the extended objects necessary for the theory to reduce to a well-defined string theory. In the second paper, we will show that the way the background metric and matter fields are affected by the orbifold singularities is encoded in the algebraic data of a particular non-split real form of the U-duality algebra.

In [19], an algebraic analysis of a certain class of orbifolds of M-theory has already been carried out in a compact version of the setup of [15]. Their work is based on a previous investigation of the relation between the moduli space of M-theory in the neighbourhood of a spacelike singularity and its possible hyperbolic billiard description [20]. For their analysis, these authors took advantage of a previous work [21] which helped establishing a dictionary between null roots of $\mathfrak{e}_{10}$ and certain Minkowskian branes and other objects of M-theory on T^{10}. Let us briefly recall this correspondence.

In [11, 22, 23], a systematic description of the relation between a subset of the positive roots of $E_{11-D|11-D}$ and BPS objects in type II string theories and M-theory has been given. In particular, they were shown to contribute to instanton corrections to the low-energy effective theory. In $D = 1$, this suggests a correspondence between certain positive real roots of $\mathfrak{e}_{10}$ and extended objects of M-theory totally wrapped in the compact space (such as Euclidean Kaluza-Klein particles, Euclidean M2 and M5-branes, and Kaluza-Klein monopoles). In the hyperbolic billiard approach to the moduli space of M-theory near a cosmological singularity, these real roots appear in exponential terms in the low-energy effective Hamiltonian of the theory [24, 18]. Such contributions behave as sharp wall potentials in the BKL limit, interrupting and reflecting the otherwise free-moving Kasner metric evolution. The latter can be represented mathematically by the inertial dynamics of a vector in the Cartan subalgebra of $\mathfrak{e}_{10}$ undergoing Weyl-reflections when it reaches the boundary of a Weyl chamber. In the low energy $11D$ supergravity limit, these sharp walls terms can be regarded as fluxes, which are changed by integer amounts by instanton effects. This description, however, is valid only in a regime where all compactification radii can all become simultaneously larger than the Planck length. In this case, the corresponding subset of positive real roots of $\mathfrak{e}_{10}$ can safely be related to instanton effect. As shown in [20], the regions of the moduli space of M-theory where this holds true are bounded by the (approximate) Kasner solution mentioned above. A proper description of these regions calls for a modification of the Kasner evolution by introducing matter, which leads to a (possibly) non-chaotic behaviour of the system at late time (or large volume). The main contribution of [21] was to give evidence that these matter contributions have a natural description in terms of imaginary roots of $\mathfrak{e}_{10}$. More precisely, these authors have shown that extended objects such as Minkowskian Kaluza-Klein particles, M2-branes, M5-branes, and Kaluza-Klein monopoles (KK7M-branes) can be related to prime isotropic imaginary roots of $\mathfrak{e}_{10}$ that, interestingly enough, are all Weyl-equivalent. These results, although derived in a compact setting, are amenable to the non-compact case [15, 18].

Ref.[19] only considers a certain class of orbifolds of M-theory, namely: $T^{10-q} \times T^q/\mathbb{Z}_2$ for $q = 1, 4, 5, 8, 9$. After orbifold projection, the residual U-duality algebra $\mathfrak{g}_{\text{inv}}$ describing the untwisted sectors of all these orbifolds was shown to possess a root lattice isomorphic to the root lattice of the over-extended hyperbolic $\mathfrak{de}_{10|10}$. However, a careful root-space analysis led the authors to the conclusion that $\mathfrak{g}_{\text{inv}}$ was actually bigger than its hyperbolic counterpart, and contained $\mathfrak{de}_{10|10}$ as a proper subalgebra. Furthermore, in the absence of flux, anomaly cancellation in such orbifolds of M-theory is known to require the insertion of 16 M$(10-q)$-branes, Kaluza-Klein particles/monopoles or other BPS objects (the $S^1/\mathbb{Z}_2$ has to be treated from a type IA point of view, where 16 D8-branes are required to compensate the charges of the two O8-planes) extending in the directions transverse to the orbifold [25, 26]. In [19], such brane configurations were shown to be nicely incorporated in the algebraic realization of the corresponding orbifolds. It was proven that the root lattice automorphism reproducing the $\mathbb{Z}_2$ action on the metric and the three-form field of the low effective M-theory action could always be rephrased in terms of a prime isotropic root, playing the rôle of the orbifold shift vector and describing precisely the transverse Minkowskian brane required for anomaly cancellation.

This construction in terms of automorphisms of the root lattice is however limited to the $\mathbb{Z}_2$ case, where, in particular, the diagonal components of the metric play no rôle. In order to treat the general $\mathbb{Z}_{n>2}$ orbifold case, we are in need of a more elaborate algebraic approach, which operates directly at the level of the generators of the algebra. In this regard, the works of Kac and Peterson on the classification of finite order automorphisms of Lie algebras, have inspired a now standard procedure [27 − 29] to determine the residual invariant subalgebra of a given finite dimensional Lie algebras, under a certain orbifold projection. This has in particular been used to study systematically the breaking patterns of the $E_8 \times E_8$ gauge symmetry of the heterotic string [28, 30]. The method is based on choosing an eigenbasis in which the orbifold charge operator can be rephrased as a Cartan preserving automorphism $\mathrm{Ad}e^{\frac{2i\pi}{n}H_\Lambda}$, where Λ is an element of the weight lattice having scalar product $(\Lambda|\theta_G) \leqslant n$ with the highest root of the algebra θ_G, and H_Λ is its corresponding Cartan element. The shift vector Λ then determines by a standard procedure the invariant subalgebra $\mathfrak{g}_{\text{inv}}$ for all $\mathbb{Z}_n$ projections. However, the dimensionality and the precise set of charges of the orbifold have to be established by other means. This is in particular necessary to isolate possible degenerate cases. Finally, this method relies on the use of extended (not affine) Dynkin diagrams, and it is not yet known how it can be generalized to affine and hyperbolic Kac-Moody algebras.

Here we adopt a novel point of view based on the observation that the action of an orbifold on the symmetry group of any theory that possesses at least global Lorentz symmetry can be represented by the rigid action of a formal rotation operator in any orbifolded plane. In algebraic terms, the orbifold charge operator will be represented by a non-Cartan preserving, finite-order automorphism acting on the appropriate complex combinations of generators. These combinations are the components of tensors in the complex basis of the orbifolded torus which diagonalize the automorphism. Thus, they reproduce the precise mapping between orbifolded generators in $\mathfrak{e}_{10|10}/\mathfrak{k}(\mathfrak{e}_{10|10})$ and charged states in the moduli space of M-theory on T^{10}. It also enables one to keep track of the reality properties of the invariant subalgebra, provided we work with a Cartan decomposition of the original U-duality algebra. This is one reason which prompted us to choose the *symmetric gauge* (in contrast to the *triangular* Iwasawa gauge) to parametrize the physical fields of the theory. For this gauge choice, the orbifold charge operator is expressible as $\prod_{\alpha\in\Delta_+} \mathrm{Ad}e^{\frac{2\pi Q_\alpha}{n}(E_\alpha - F_\alpha)}$, for $E_\alpha - F_\alpha \in \mathfrak{k}(\mathfrak{e}_{10|10})$, and Δ_+ a set of positive roots reproducing the correct orbifold charges $\{Q_\alpha\}$. The fixed point subalgebra $\mathfrak{g}_{\text{inv}}$ is then obtained by truncating to the $Q_\alpha = kn$ sector, $k \in \mathbb{Z}$.

This method is general and can in principle be applied to the U-duality symmetry of any orbifolded supergravities and their M-theory limits. In this paper, we will restrict ourselves to the $\mathfrak{g}^U = \mathfrak{e}_{11-D|11-D}$ U-duality chain for $8 \leqslant D \leqslant 1$. We will also limit our detailed study to a few illustrative examples of orbifolds, namely: $T^{11-D-q} \times T^q/\mathbb{Z}_{n>2}$ for $q = 2, 4, 6$ and $T^{11-D-q} \times T^q/\mathbb{Z}_2$ for $q = 1, \dots, 9$. In the $\mathbb{Z}_2$ case, we recover, for $q = 1, 4, 5, 8, 9$, the results of [19]. In the other cases, the results are original and lead, for $D = 1$, to several examples where we conjecture that $\mathfrak{g}_{\text{inv}}$ is obtained by modding out either a Borcherds algebra or an indefinite (not affine) Kac-Moody algebra, by its centres and derivations. As a first check of this conjecture, we study in detail the $T^8 \times T^2/\mathbb{Z}_n$ case, and verify its validity up to level $l = 6$, investigating with care the splitting of the multiplicities of the original $\mathfrak{e}_{10}$ roots under the orbifold projection. We also show that the remaining cases can be treated in a similar fashion. From a different perspective [31, 32], truncated real super-Borcherds algebras have been shown to arise already as more general symmetries of various supergravities expressed in the doubled formalism and compactified on square tori to $D = 3$. Our work, on the other hand, gives other explicit examples of how Borcherds algebras may appear as the fixed-point subalgebras of a hyperbolic Kac-Moody algebra under a finite-order automorphism.

Subsequently, we engineer the relation between our orbifolding procedure which relies on finite order non-Cartan preserving automorphism, and the formalism of Kac-Peterson [33]. We first show that there is a new *primed* basis of the algebra in which one can derive a class of shift vectors for each orbifold we have considered. Then, we prove that these vectors are, for a given n, conjugate to the shift vector expected from the Kac-Peterson formalism. We show furthermore that, in the primed basis, every such class contains a positive root of $\mathfrak{e}_{10}$, which can serve as class representative. This root has the form $\Lambda'_n + n\tilde{\delta}'$, where $\tilde{\delta}'$ is in the same orbit as the null root δ' of $\mathfrak{e}_9$ under the Weyl group of $SL(10)$, and is minimal, in the sense that Λ'_n is the minimal weight leading to the required set of orbifold charges. We then list all such class representatives for all orbifolds of the type $T^{q_1}/\mathbb{Z}_{n_1} \times \cdots \times T^{q_m}/\mathbb{Z}_{n_m}$ with $\sum_{i=1}^{m} q_i \leqslant 10$, where the $\mathbb{Z}_{n_i}$ actions act independently on each T^2 subtori.

In particular, for the $T^{10-q} \times T^q/\mathbb{Z}_2$ orbifolds of M-theory with $q = 2, 3, 6, 7$ that were not considered in [19], we find that a consistent physical interpretation requires to consider them in the bosonic M-theory that descend to type 0A strings. In such cases, we find class representatives that are either positive real roots of $\mathfrak{e}_{10}$, or positive non-isotropic imaginary roots of norm -2. We then show that these roots are related to the twisted sectors of some particular non-supersymmetric type 0' orientifolds carrying magnetic fluxes. Performing the reduction to type 0A theory and T-dualizing appropriately, we actually find that these roots of $\mathfrak{e}_{10}$ descend to magnetized D9-branes in type 0' orientifolds carrying $(2\pi)^{-[q/2]} \int \mathrm{Tr}\, \underbrace{F \wedge .. \wedge F}_{[q/2]}$ units of flux. This gives a partial characterization of open strings twisted sectors in non-supersymmetric orientifolds in terms of roots of $\mathfrak{e}_{10}$. Moreover, the fact that these roots can be identified with Minkowskian D-branes even though none of them is prime isotropic, calls for a more general algebraic characterization of Minkowskian objects than the one propounded in [19]. A new proposal supported by evidence from the $\mathbb{Z}_{n>2}$ case will be presented in Section 10.2.

More precisely, the orbifolds of M-theory mentioned above descend to orientifolds of type 0A string theory by reducing on one direction of the orbifolded torus for q even, and on one direction outside the torus for q odd. T-dualizing to type 0B, we find cases similar to those studied in [34, 35], where specific configurations of D-branes and D'-branes were used to cancel the two 10-form RR tadpoles. Here, in contrast, we consider a configuration in which the branes are tilted with respect to the orientifold planes in the 0A theory. This setup is T-dual to a type 0' orientifold with magnetic fluxes coupling to the electric charge of a D$(10-q)$-brane embedded in the space-filling D9-branes in the spirit of [36, 37]. In this perspective, the aforementioned roots of $\mathfrak{e}_{10}$ which determine the $\mathbb{Z}_2$ action also possess a dual description in terms of tilted D-branes of type 0A string theory. In the original 11-dimensional setting, these roots are related to exotic objects of M-theory and thereby provide a proposal for the M-theory origin of such configurations.

Finally, we will also comment on the structure of the $\mathfrak{e}_{10}$ roots that appear as class representatives for shift vectors of $\mathbb{Z}_n$ orbifolds of M-theory, and hint at the kind of flux configurations these roots could be associated to.

2. Generalized Kac-Moody algebras

In this section, we introduce recent mathematical constructions from the theory of infinite-dimensional Lie algebras. Indeed, it is well-known in Lie theory that fixed-point subalgebras of infinite-dimensional Lie algebras under certain algebra automorphisms are often interesting mathematical objects in their own right and might have quite different properties. Of particular interest here is the fact that fixed-point subalgebras of Kac-Moody algebras are not necessarily Kac-Moody algebras, but can belong to various more general classes of algebras like extended affine Lie algebras [38 − 40], generalized Kac-Moody algebras [41 − 43], Slodowy intersection matrices [44] or Berman's generalized intersection matrices [45]. Indeed, invariant U-duality symmetry subalgebras for orbifolds of M-theory are precisely fixed-point subalgebras under a finite-order automorphism and can be expected (at least in the hyperbolic and Lorentzian cases) to yield algebras that are beyond the realm of Kac-Moody algebras.

2.1 Central extensions of Borcherds algebras

Since they are particularly relevant to our results, we will focus here on the so-called generalized Kac-Moody algebras, or GKMAs for short, introduced by Borcherds in [41] to extend the Kac-Moody algebras construction to infinite-dimensional algebras with imaginary simple roots. We define here a number of facts and notations about infinite-dimensional Lie algebras which we will need in the rest of the paper, starting from very general considerations and then moving to more particular properties. This will eventually prompt us to refine the approach to GKMAs with a degenerate Cartan matrix, by providing, in particular, a rigorous definition of how scaling operators should by introduced in this case in accordance with the general definition of GKMAs (see for instance Definitions 2.3 and 2.4 below). This has usually been overlooked in the literature, but turns out to be crucial for our analysis of fixed point subalgebras of infinite KMAs under a finite order automorphism, which occur, as we will see, as hidden symmetries of the untwisted sector of M-theory under a given orbifold.

In this perspective, we start by defining the necessary algebraic tools. Let $\mathfrak{g}$ be a (possibly infinite-dimensional) Lie algebra possessing a Cartan subalgebra $\mathfrak{h}$ (a complex nilpotent subalgebra equal to its normalizer) which is *splittable*, in other words, the action of ad$H$ on $\mathfrak{g}$ is trigonalizable $\forall H \in \mathfrak{h}$. The derived subalgebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ then possesses an r-dimensional Cartan subalgebra $\mathfrak{h}' = \mathfrak{g}' \cap \mathfrak{h}$ spanned by the basis $\Pi^\vee = \{H_i\}_{i\in I}$, with indices valued in the set $I = \{i_1, .., i_r\}$.

We denote by $\mathfrak{h}'^*$ the space dual to $\mathfrak{h}'$. It has a basis formed by r linear functionals (or 1-forms) on $\mathfrak{h}'$, the simple roots of $\mathfrak{g}$: $\Pi = \{\alpha_i\}_{i\in I}$. Suppose we can define an indefinite scalar product: $(\alpha_i|\alpha_j) = a_{ij}$ for some real $r \times r$ matrix a, then:

Definition 2.1 The matrix a is called a generalized symmetrized Cartan matrix, if it satisfies the conditions:

i) $a_{ij} = a_{ji}\,,\ \forall i,j \in I.$

ii) a has no zero column.

iii) $a_{ij} \leq 0\,$, for $i \neq j$ and $\forall i,j \in I$.

iv) $\left.\begin{array}{l}\text{if } a_{ii} \neq 0\ :\ 2\frac{a_{ij}}{a_{ii}} \\ \text{if } a_{ii} = 0\ :\ \quad a_{ij}\end{array}\right\} \in \mathbb{Z}\,$, for $i \neq j$ and $\forall i,j \in I$.

From integer linear combinations of simple roots, one constructs the root lattice $Q = \sum_{i\in I} \mathbb{Z}\alpha_i$. The scalar product $(\,|\,)$ is then extended by linearity to the whole $Q \subset \mathfrak{h}'^*$. Furthermore, by defining fundamental weights $\{\Lambda^i\}_{i\in I}$ satisfying $(\Lambda^i|\alpha_j) = \delta^i{}_j\,,\ \forall i,j \in I$, we introduce a duality relation with respect to the root scalar product. Then, from the set $\{\Lambda^i\}_{i\in I}$ we define the lattice of integral weights $P = \sum_{i\in I} \mathbb{Z}\Lambda^i$ dual to Q, such that $Q \subseteq P$.

Let us introduce the duality isomorphism ν: $\mathfrak{h}' \to \mathfrak{h}'^*$ defined by

$$\nu(H_i) = \begin{cases} 2\frac{\alpha_i}{|a_{ii}|} & ,\ \text{if } a_{ii} \neq 0\,. \\ \alpha_i & ,\ \text{if } a_{ii} = 0\,, \end{cases} \quad . \tag{2.1}$$

We may now promote the scalar product $(\alpha_i|\alpha_j) = a_{ij}$ to a symmetric bilinear form B on $\mathfrak{h}'$ through:

$$b_{ij} = B(H_i, H_j) = (\nu(H_i)|\nu(H_j))\,,\ \forall i,j \in I\,.$$

Suppose next that the operation $\mathrm{ad}(H)$ is diagonalizable $\forall H \in \mathfrak{h}$, from which we define the following:

Definition 2.2 We call root space an eigenspace of $\mathrm{ad}(H)$ defined as

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}\,|\,\mathrm{ad}(H)X = \alpha(H)X,\ \forall H \in \mathfrak{h}\} \tag{2.2}$$

which defines the root system of $\mathfrak{g}$ as $\Delta(\mathfrak{g},\mathfrak{h}) = \{\alpha \neq 0\,|\,\mathfrak{g}_\alpha \neq \{0\}\}$, depending on the choice of basis for $\mathfrak{h}$.

The multiplicities attached to a root $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$ are then given by $m_\alpha = \dim \mathfrak{g}_\alpha$. As usual, the root system splits into a positive root system and a negative root system. The positive root system is defined as

$$\Delta_+(\mathfrak{g},\mathfrak{h}) = \Big\{\alpha \in \Delta(\mathfrak{g},\mathfrak{h})\,|\,\alpha = \sum_{i\in I} n_i\alpha_i, \text{with } n_i \in \mathbb{N}\,, \forall i \in I\Big\}$$

and the negative root one as $\Delta_-(\mathfrak{g},\mathfrak{h}) = -\Delta_+(\mathfrak{g},\mathfrak{h})$, so that $\Delta(\mathfrak{g},\mathfrak{h}) = \Delta_+(\mathfrak{g},\mathfrak{h}) \cup \Delta_-(\mathfrak{g},\mathfrak{h})$. We call $\mathrm{ht}(\alpha) = \sum_{i\in I} n^i$ the height of α. From now on, we shall write $\Delta \equiv \Delta(\mathfrak{g},\mathfrak{h})$ for economy, and restore the full notation $\Delta(\mathfrak{g},\mathfrak{h})$ or partial notation $\Delta(\mathfrak{g})$ when needed.

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Finally, since $(\alpha|\alpha)$ is bounded above on Δ, α is called real if $(\alpha|\alpha) > 0$, isotropic imaginary if $(\alpha|\alpha) = 0$ and (non-isotropic) imaginary if $(\alpha|\alpha) < 0$. Real roots always have multiplicity one, as is the case for finite-dimensional semi-simple Lie algebras, while (non-simple) isotropic roots have a multiplicity equal to $\mathrm{rk}a(\hat{\mathfrak{g}})$ for some affine subalgebra $\hat{\mathfrak{g}} \subset \mathfrak{g}$, while (non-simple) non-isotropic imaginary roots can have very big multiplicities.

Generalized Kac-Moody algebras are usually defined with all *imaginary simple* roots of multiplicity one, as well. One could in principle define a GKMA with simple roots of multiplicities bigger than one, but then the algebra would not be completely determined by its generalized Cartan matrix. In this case, one would need yet another matrix with coefficients specifying the commutation properties of all generators in the same simple root space. Here, we shall not consider this possibility further since it will turn out that all fixed point subalgebras we will be encountering in the framework of orbifold compactification of $11D$ supergravity and M-theory possess only isotropic simple roots of multiplicity one.

We now come to specifying the rôle of central elements and scaling operators in the case of GKMAs with degenerate generalized Cartan matrix.

Definition 2.3 If the matrix a does not have maximal rank, define the centre of $\mathfrak{g}$ as $\mathfrak{z}(\mathfrak{g}) = \{c \in \mathfrak{h} \,|\, B(H_i, c) = 0\,,\, \forall i \in I\}$. In particular, if $l = \dim \mathfrak{z}(\mathfrak{g})$, one can find l linearly independent null root lattice vectors $\{\delta_i\}_{i=1,..,l}$ (possibly roots, but not necessarily) satisfying $(\delta_i|\nu(H_j)) = 0\,,\, \forall i = 1, \dots, l,\, \forall j \in I$. One then defines l linearly independent Cartan generators $\{d_i\}_{i=1,..,l}$ with $d_i \in \mathfrak{h}/\mathfrak{h}'$ thus extending the bilinear form B to the whole Cartan algebra $\mathfrak{h}$ as follows:

- $B(c_i, d_j) = \delta_{ij}\,,\, \forall i, j = 1, .., l\,.$
- $B(d_i, d_j) = 0\,,\, \forall i, j = 1, .., l\,.$
- $B(H, d_i) = 0\,,\, \forall i = 1, .., l$ and for $H \in \mathfrak{h}'/\mathrm{Span}\{c_i\}_{i=1,..,l}\,.$

Then, we have $\mathrm{rk}(a) = r - l$ and $\dim \mathfrak{h} = r + l$.

This definition univocally fixes the i-th level k_i of all roots $\alpha \in \Delta$ to be $k_i = B(\nu^{-1}(\alpha), d_i)$, using the decomposition of $\nu^{-1}(\alpha)$ on orthogonal subspaces in $\mathfrak{h}' = (\mathfrak{h}'/\mathrm{Span}\{c_1, \dots, c_l\}) \oplus \mathrm{Span}\{c_1\} \oplus \cdots \oplus \mathrm{Span}\{c_n\}$.

We are now ready to define a GKMA by its commutation relations. Definitions of various levels of generality exist in the literature, but we choose one that is both convenient (though seemingly complicated) and sufficient for our purpose, neglecting the possibility that $[E_i, F_j] \neq 0$ for $i \neq j$ (see, for example, [46, 47] for such constructions), but taking into account the possibility of degenerate Cartan matrices, a generic feature of the type of GKMA we will be studying later on in this paper.

Definition 2.4 The universal generalized Kac-Moody algebra associated to the Cartan matrix a is the Lie algebra defined by the following commutation relations (Serre-Chevalley basis) for the simple root generators $\{E_i, F_i, H_i\}_{i\in I}$:

1. $[E_i, F_j] = \delta_{ij} H_i, \qquad [H_i, H_j] = [H_i, d_k] = 0\,, \qquad \forall i, j \in I, \quad k = 1, .., l\,.$

2. $[H_i, E_j] = \begin{cases} \frac{2a_{ij}}{|a_{ii}|} E_j \,, \text{ if } a_{ii} \neq 0 \\ a_{ij} E_j \,, \text{ if } a_{ii} = 0 \end{cases} , \qquad [H_i, F_j] = \begin{cases} -\frac{2a_{ij}}{|a_{ii}|} F_j \,, \text{ if } a_{ii} \neq 0 \\ -a_{ij} F_j \,, \text{ if } a_{ii} = 0 \end{cases} , \qquad \forall i, j \in I \,.$

3. If $a_{ii} > 0$: $\quad (\mathrm{ad} E_i)^{1-2\frac{a_{ij}}{a_{ii}}} E_j = 0, \qquad (\mathrm{ad} F_i)^{1-2\frac{a_{ij}}{a_{ii}}} F_j = 0 \,, \qquad \forall i, j \in I \,.$

4. $\forall i, j \in I$ such that $a_{ii} \leq 0$, $a_{jj} \leq 0$ and $a_{ij} = 0$:[1] $\qquad [E_i, E_j] = 0, \qquad [F_i, F_j] = 0 \,,$

5. $[d_i, [E_{j_1}, [E_{j_2}, .., E_{j_n}]..]] = k_i [E_{j_1}, [E_{j_2}, .., E_{j_n}]..] \,,$
   $[d_i, [F_{j_1}, [F_{j_2}, .., F_{j_n}]..]] = -k_i [F_{j_1}, [F_{j_2}, .., F_{j_n}]..] \,,$
   where k_i is the $i-th$ level of $\alpha = \alpha_{j_1} + \ldots + \alpha_{j_n}$, as defined above.

Since a generalized Kac-Moody algebra can be graded by its root system as: $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, the indefinite scalar product B can be extended to an $\mathrm{ad}(\mathfrak{g})$-invariant bilinear form satisfying: $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ except if $\alpha + \beta = 0$, which we call the generalized Cartan-Killing form. It can be fixed uniquely by the normalization

$$B(E_i, F_j) = \begin{cases} \frac{2}{|a_{ii}|} \delta_{ij} \,, & \text{if } a_{ii} \neq 0 \\ \delta_{ij} \,, & \text{if } a_{ii} = 0 \end{cases} ,$$

on generators corresponding to simple roots. Then $\mathrm{ad}(\mathfrak{g})$-invariance naturally implies: $B(H_i, H_j) = (\nu(H_i) | \nu(H_j))$.

The GKMA $\mathfrak{g}$ can be endowed with an antilinear Chevalley involution ϑ_C acting as $\vartheta_C(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ and $\vartheta_C(H) = -H$, $\forall H \in \mathfrak{h}$, whose action on each simple root space $\mathfrak{g}_{\alpha_i}$ is defined as as usual as $\vartheta_C(E_i) = -F_i$, $\forall i \in I$. The Chevalley involution extends naturally to the whole algebra $\mathfrak{g}$ by linearity, for example:

$$\vartheta_C([E_i, E_j]) = [\vartheta_C(E_j), \vartheta_C(E_j)] = [F_i, F_j] \,. \tag{2.3}$$

This leads to the existence of an almost positive-definite contravariant form $B_{\vartheta_C}(X, Y) = -B(\vartheta_C(X), Y)$. More precisely, it is positive-definite everywhere outside $\mathfrak{h}$.

Note that there is another standard normalization, the Cartan-Weyl basis, given by:

$$e_{\alpha_i} = \begin{cases} \sqrt{\frac{|a_{ii}|}{2}} E_i \,, & \text{if } a_{ii} \neq 0 \\ E_i \,, & \text{if } a_{ii} = 0 \end{cases} , \qquad f_{\alpha_i} = \begin{cases} \sqrt{\frac{|a_{ii}|}{2}} F_i \,, & \text{if } a_{ii} \neq 0 \\ F_i \,, & \text{if } a_{ii} = 0 \end{cases} ,$$

$$h_{\alpha_i} = \begin{cases} \frac{|a_{ii}|}{2} H_i \,, & \text{if } a_{ii} \neq 0 \\ H_i \,, & \text{if } a_{ii} = 0 \end{cases} ,$$

and characterized by: $B(e_\alpha, f_\alpha) = 1$, $\forall \alpha \in \Delta_+(\mathfrak{g})$.

We will not use this normalization here, but we will instead write the Cartan-Weyl relations in a Chevalley-Serre basis, as follows:

[1] Note that there is no a priori limit to the number of times one can commute the generator E_i for α_i imaginary with any other generator E_j in case $a_{ij} \neq 0$.

Definition 2.5 For all $\alpha \in \Delta_+(\mathfrak{g})$ introduce $2m_\alpha$ generators: E^a_α and F^a_α, $a = 1,..,m_\alpha$. Generators corresponding to roots of height ± 2 are defined as:

$$E_{\alpha_i+\alpha_j} = \mathcal{N}_{\alpha_i,\alpha_j}[E_i, E_j], \quad F_{\alpha_i+\alpha_j} = \mathcal{N}_{-\alpha_i,-\alpha_j}[F_i, F_j],\ \forall i,j \in I\,,$$

for a certain choice of structure constants $\mathcal{N}_{\alpha_i,\alpha_j}$. Then, higher heights generators are defined recursively in the same way through:

$$[E^a_\alpha, E^b_\beta] = \sum_c (\mathcal{N}_{\alpha,\beta})^{ab}{}_c E^c_{\alpha+\beta}\,. \tag{2.4}$$

The liberty of choosing the structure constants is of course limited by the anti-commutativity of the Lie bracket: $(\mathcal{N}_{\alpha,\beta})^{ab}{}_c = -(\mathcal{N}_{\beta,\alpha})^{ba}{}_c$ and the Jacobi identity, from which we can derive several relations. Among these, the following identity, valid for finite-dimensional Lie algebras, will be useful for our purposes:

$$\mathcal{N}_{\alpha,\beta}\mathcal{N}_{-\alpha,-\beta} = -(p+1)^2,\ p \in \mathbb{N}\,,\ \text{s.t. } \{\beta - p\alpha, \ldots, \beta+\alpha\} \subset \Delta(\mathfrak{g},\mathfrak{h})\,.$$

Note that this relation can be generalized to the infinite-dimensional case if one chooses the bases of root spaces $\mathfrak{g}_\alpha$ with $m_\alpha > 1$ in a particular way such that there is no need for a sum in (2.4). Imposing in addition $(\mathcal{N}_{\alpha,\beta})^{ab}{}_c = -(\mathcal{N}_{-\alpha,-\beta})^{ab}{}_c$ gives sign conventions compatible with $\vartheta_C(E^a_\alpha) = -F^a_\alpha\,,\ \forall\alpha \in \Delta_+(\mathfrak{g}), a = 1\ldots, m_\alpha$, not only for simple roots. In the Serre-Chevalley normalization, this furthermore ensures that: $\mathcal{N}_{\alpha,\beta} \in \mathbb{Z}$, $\forall\alpha,\beta \in \Delta$. Here lies our essential reason for sticking to this normalization, and we will follow this convention throughout the paper. In the particular case of *simply-laced* semi-simple Lie algebras, we always have $p = 0$, and we can choose $\mathcal{N}_{\alpha,\beta} = \pm 1\,,\ \forall\alpha,\beta \in \Delta$ (note, however, that this is not true for infinite-dimensional simply-laced algebras).

Another important consequence of the Jacobi identity in the finite case, which will turn out to be useful is the following relation

$$\mathcal{N}_{\alpha,-\beta} = \mathcal{N}_{\beta-\alpha,\alpha} \quad \forall\alpha,\beta \in \Delta\,.$$

2.2 Kac-Moody algebras as a special case of GKMA

Standard symmetrized Kac-Moody algebras (KMA) can be recovered from the preceding section by imposing $a_{ii} > 0\,,\ \forall i \in I$ in all the above definitions. In addition, one usually rephrases the dual basis $\Pi^\vee$ in terms of coroots, by setting $\alpha^\vee_i \equiv H_i$. Their image under the duality isomorphism reads

$$\nu(\alpha^\vee_i) = \frac{2}{(\alpha_i|\alpha_i)}\alpha_i\,,\ \forall i \in I\,,$$

so that instead of the symmetrized Cartan matrix a, one generally resorts to the following non-symmetric version, defined as a realization of the triple $\{\mathfrak{h}, \Pi, \Pi^\vee\}$ with $\Pi^\vee = \{\alpha^\vee_i\}_{i\in I} \subset \mathfrak{h}^*$:

$$A_{ij} = \frac{2a_{ij}}{a_{ii}} \equiv \langle\alpha^\vee_i, \alpha_j\rangle\,. \tag{2.5}$$

where the duality bracket on the RHS represents the standard action of the one-form α_j on the vector $\alpha_i^\vee$.

The matrix a is then called the symmetrization of the (integer) Cartan matrix A. As a consequence of having introduced l derivations in Definition 2.3, the contravariant form $B_{\vartheta_C}(.,.)$ now becomes non-degenerate on the whole of $\mathfrak{g}$, even in the case of central extensions of multi-loop algebras, which are the simplest examples of extended affine Lie algebra (EALA, for short).

For the following, we need to introduce the Weyl group of $\mathfrak{g}$ as

Definition 2.6 The Weyl group of $\mathfrak{g}$, denoted $W(\mathfrak{g})$, is the group generated by all reflections mapping the root system into itself:

$$\begin{aligned} r_\alpha\, :\, \Delta(\mathfrak{g}) &\to \Delta(\mathfrak{g}) \\ \beta &\mapsto \beta - \langle\alpha^\vee,\beta\rangle\alpha\,. \end{aligned}$$

The set $\{r_{i_1},..,r_{i_r}\}$, where $r_i \doteq r_{\alpha_i}$ are called the fundamental reflections, is a basis of $W(\mathfrak{g})$. Since $r_i^{-1} = r_{-\alpha_i}$, $W(\mathfrak{g})$ is indeed a group.

The real roots of any finite Lie algebra or KMA can then be defined as being Weyl conjugate to a simple root. In other words, $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$ is real if $\exists w \in W(\mathfrak{g})$ such that $\alpha = w(\alpha_i)$ for $i \in I$ and $\mathfrak{g}$ *is* a KMA.

A similar formulation exists for imaginary roots of a KMA, which usually turns out to be useful for determining their multiplicities, namely (see [48]):

Theorem 2.1 *Let $\alpha = \sum_{i\in I} k_i\alpha_i \in Q\backslash\{0\}$ have compact support on the Dynkin diagram of $\mathfrak{g}$, and set:*

$$K = \{\alpha\,|\,\langle\alpha_i^\vee,\alpha\rangle \leqslant 0\,,\ \forall i \in I\}\,,$$

Then denoting by Δ_{im} the set of imaginary roots of $\mathfrak{g}$, we have:

$$\Delta_{im}(\mathfrak{g}) = \bigcup_{w\in W(\mathfrak{g})} w(K)\,.$$

It follows from Theorem 2.1 that, in the affine case, every isotropic root α is Weyl-equivalent to $n\delta$ (with $\delta = \alpha_0 + \theta$ the null root) for some $n \in \mathbb{Z}^*$, which is another way of showing that all such roots have multiplicity $m_\alpha = r$. All isotropic roots which are Weyl-equivalent to δ are usually called *prime* isotropic. Note, finally, that statements similar to Theorem 2.1 holds for non-isotropic imaginary roots of hyperbolic KMAs. For instance, all positive roots with $(\alpha|\alpha) = -2$ can in this case be shown to be Weyl-equivalent to Λ^0, the weight dual to the extended root α_0.

Intersection matrix algebras are even more general objects that allow for positive non-diagonal elements in the Cartan matrix. Slodowy intersection matrices allow such positive diagonal metric elements, while Berman generalized intersection matrices give the most general framework by allowing imaginary simple roots, as well, as in the case of Borcherds algebras. Such more complicated algebras will not appear in the situations considered in this paper, but it is not impossible that they could show up in applications of the same methods to different situations.

2.3 A comment on $\mathfrak{sl}(10)$ representations in $\mathfrak{e}_{10}$ and their outer multiplicity

Of particular significance for Kac-Moody algebras beyond the affine case are of course the root system and the root multiplicities, which are often only partially known. Fortunately, in the case of $\mathfrak{e}_{10}$, we can rely on the work in [49−51] to obtain information about a large number of low-level roots, enough to study $\mathbb{Z}_n$ orbifolds up to $n = 6$. These works rely on decomposing Lorentzian algebras in representations of a certain finite subalgebra. However, the set of representations is not exactly isomorphic to the root system (modulo Weyl reflections). Indeed, the multiplicity of a representation in the decomposition is in general smaller than the multiplicity of the root corresponding to its highest weight vector. Typically, the m-dimensional vector space corresponding to a root of (inner) multiplicity m will be split into subspaces of several representations of the finite subalgebra. Typically, a root α of multiplicity $m_\alpha > 1$ will appear $n_{\rm o}(\alpha)$ times as the highest weight vector of a representation, plus several times as a weight of other representations. The number $n_{\rm o}(\alpha)$ is called the outer multiplicity, and can be 0. For a representation $\mathcal{R}$ of $\mathfrak{g}$ it shall be denoted by a subscript as: $\mathcal{R}_{[n_{\rm o}]}$ when needed. Even though the concept of outer multiplicity is of minor significance for our purpose, it is important to understand the mapping between the results of [52, 50], based on representation of finite subalgebras, and ours, which focuses on tensorial representations with definite symmetry properties.

3. Hidden symmetries in M-theory: the setup

As a start, we first review some basic facts about hidden symmetries of $11D$ supergravity and ultimately M-theory, ranging from the early non-linear realizations of toroidally compactified $11D$ supergravity [6, 7] to the conjectured hyperbolic $\mathfrak{e}_{10}$ hidden symmetry of M-theory.

Then in Sections 3.1-3.5, we do a synthesis of the algebraic approach to U-duality symmetries of $11D$ supergravity on T^q and the moduli space of M-theory on T^{10}, presenting in detail the physical material and mathematical tools that we will need in the subsequent sections, and justify our choice of parametrization for the coset element (algebraic field strength) describing the physical fields of the theory. The reader familiar with these topics may of course skip the parts of this presentation he will judge too detailed.

The global $E_{11-D|11-D}$ symmetry of classical $11D$ supergravity reduced on T^{11-D} for $10 \leqslant D \leqslant 3$ can be best understood as arising from a simultaneous realization of the linear non-perturbative symmetry of the supergravity Lagrangian where no fields are dualized and the perturbative T-duality symmetry of type IIA string theory appearing in $D = 10$ and below. Actually, the full $E_{11-D|11-D}$ symmetry has a natural interpretation as the closure of both these groups, up to shift symmetries in the axionic fields.

Type IIA string theory compactified on T^{10-D} enjoys a $SO(10-D, 10-D, \mathbb{Z})$ symmetry[2] which is valid order by order in perturbation theory. So, restricting to massless

[2]We consider $SO(10-D, 10-D)$ instead of $O(10-D, 10-D)$, as is sometimes done, because the elements of $O(10-D, 10-D)$ connected to $-\mathbb{1}$ flip the chirality of spinors in the type IIA/B theories. As such, this subset of elements is not a symmetry of the R-R sector of the type IIA/B supergravity actions, but dualities which exchange both theories.

scalars arising from T-duality in $D \leqslant 8$, all inequivalent quantum configurations of the scalar sector of the bosonic theory are described by the moduli space

$$\mathcal{M}_D = SO(10-D, 10-D, \mathbb{Z}) \backslash SO(10-D, 10-D)/(SO(10-D) \times SO(10-D)),$$

where the left quotient by the arithmetic subgroup corrects the over-counting of perturbative string compactifications. In contrast to the NS-NS fields B_2 and $g_{\mu\nu}$ which, at the perturbative level, couple to the string worldsheet, the R-R fields do so only via their field strength. So a step towards U-duality can be achieved by dualizing the R-R fields while keeping the NS-NS ones untouched. It should however be borne in mind that such a procedure enhances the T-duality symmetry only when dualizing a field strength to an equal or lower rank one. Thus, Hodge-duals of R-R fields start playing a rôle when $D \leqslant 8$, those of NS-NS fields when $D \leqslant 6$. However, when perturbative symmetries are concerned, we will not dualize NS-NS fields.

This enlarged T-duality symmetry can be determined by identifying its discrete Weyl group $W(D_{10-D})$ [8], which implements the permutation of field strengths required by electric-magnetic duality. In $D \leqslant 8$ it becomes now necessary to dualize R-R field strengths in order to form Weyl-group multiplets. This results in 2^{9-D} R-R axions, all exhibiting a shift symmetry, that enhances the T-duality group to:

$$\widetilde{G} = SO(10-D, 10-D) \ltimes \mathbb{R}^{2^{9-D}}, \tag{3.1}$$

the semi-direct product resulting from the fact that the R-R axions are now linearly rotated into one another under T-duality. The (continuous) scalar manifold is now described by the coset $\widetilde{G}/SO(10-D) \times SO(10-D)$, whose dimension matches the total number of scalars if we include the duals of R-R fields only. The symmetry (3.1) can now be enlarged to accomodate non-perturbative generators, leading to the full global symmetry $E_{11-D|11-D}$. However, this can only be achieved without dualizing the NS-NS fields in the range $9 \geqslant D \geqslant 7$. When descending to lower dimensions, indeed, the addition of non-perturbative generators rotating R-R and NS-NS fields into one another forces the latter to be dualized.

To evade this problem arising in low dimensions, we might wish to concentrate instead on the global symmetry of the $11D$ supergravity Lagrangian for $D \leqslant 9$, whose scalar manifold is described by the coset

$$GL(11-D) \ltimes \mathbb{R}^{(11-D)!/((8-D)!3!)}/O(11-D).$$

The corresponding group $G_{SG} = GL(11-D) \ltimes \mathbb{R}^{(11-D)(10-D)(9-D)/6}$ encodes the symmetry of the totally undualized theory including the $(11-D)(10-D)(9-D)/6$ shift symmetries coming from the axions produced by toroidal compactification of the three-form C_3. Again, the semi-direct product reflects the fact that these axions combine in a totally antisymmetric rank three representation of $GL(11-D)$. Since NS-NS and R-R fields can be interchanged by $GL(11-D)$, the arithmetic subgroup of $GL(11-D) \ltimes \mathbb{R}^{(11-D)(10-D)(9-D)/6}$ constitutes an acceptable non-perturbative symmetry of type II superstring theory in $D \leqslant 9$. The price to pay in this case is to sacrifice T-duality, since the subgroup of the linear group preserving the NS-NS and R-R sectors separately is never big enough to accomodate $SO(10-D,10-D)$.

Eventually, the full non-perturbative symmetry $E_{11-D|11-D}$ can only be achieved when both NS-NS and R-R fields are dualized, and may be viewed as the closure of its $GL(11-D)$ and $SO(10-D,10-D)$ subgroups. However, the number of shift symmetries in this fully dualized version of the theory is given by $\{3,6,10,16,27,44\}$ for $8 \geqslant D \geqslant 3$. Since, in $D \leqslant 5$, these numbers are always smaller or equal to $(11-D)(10-D)(9-D)/6$ and 2^{9-D}, neither $\widetilde{G}$ nor G_{SG} are subgroups of $E_{11-D|11-D}$ in low dimensions.

This exceptional symmetry is argued to carry over, in its discrete version, to the full quantum theory. Typically, the conjectured U-duality group of M-theory on T^{11-D} can be rephrased as the closure

$$E_{11-D|11-D}(\mathbb{Z}) = SO(10-D,10-D,\mathbb{Z})\bar{\times}SL(11-D,\mathbb{Z})\,, \tag{3.2}$$

where the first factor can be viewed as the perturbative T-duality symmetry of IIA string theory, while the second one is the modular group of the torus T^{11-D}. In $D=9$, the latter can be rephrased in type IIB language as the expected S-duality symmetry.

The moduli space of M-theory on T^{11-D} is then postulated to be

$$\mathcal{M}_D = E_{11-D|11-D}(\mathbb{Z})\backslash E_{11-D|11-D}/K(E_{11-D|11-D})\,. \tag{3.3}$$

It encodes both the perturbative NS-NS electric p-brane spectrum and the spectrum of non-perturbative states, composed of the magnetic dual NS-NS $(9-D-p)$-branes and the R-R D-branes of IIA theory for $10 \leqslant D \leqslant 3$.

In dimensions $D<3$, scalars are dual to themselves, so no more enhancement of the U-duality group is expected from dualization. However, an enlargement of the hidden symmetry of the theory is nevertheless believed to occur through the affine extension $E_{9|10}(\mathbb{Z})$ in $D=2$, the over-extended $E_{10|10}(\mathbb{Z})$, generated by the corresponding hyperbolic KMA, in $D=1$, and eventually the very-extended $E_{11}(\mathbb{Z})$ in the split form, whose KMA is Lorentzian, for the totally compactified theory.

Furthermore, there is evidence that the latter two infinite-dimensional KMAs are also symmetries of unreduced $11D$ supergravity [53, 54], viewed as a non-linear realization (in the same spirit as the Monstruous Moonshine [55] has been conjectured to be a symmetry of uncompactified string theory) and are believed to be more generally symmetries of uncompactified M-theory itself [56].

3.1 The exceptional E_r serie: conventions and useful formulæ

Before going into more physical details, we need to introduce a few mathematical properties of the U-duality groups and their related algebras. To make short, we will denote $G^U \doteq \mathrm{Split}(E_{11-D})$, for $D=1,\ldots,10$, and their defining Lie or Kac-Moody algebras $\mathfrak{g}^U \doteq \mathrm{Split}(\mathfrak{e}_{11-D})$.

Except in $D=10,9$, the exceptional serie E_r, with $r=11-D$, possesses a physical basis for roots and dual Cartan generators:

Definition 3.1 Let the index set of Definition 2.1 be chosen as $I = 9-r, \ldots, 8$, for $3 \leq r \leq 10$ [3], then in the *physical basis* of $\mathfrak{h}'^*$: $\{\varepsilon_{11-r} \doteq (1,0,\ldots,0), \ldots, \varepsilon_{10} \doteq (0,\ldots,0,1)\}$, the set $\Pi = \{\alpha_{9-r}, \ldots, \alpha_8\}$ of simple roots of the semi-simple KMAs $\mathfrak{e}_r$ reads:

$$\begin{aligned} \alpha_{9-r} &= \varepsilon_{11-r} - \varepsilon_{12-r} \;=\; (1,-1,0,\ldots,0)\,, \\ &\vdots \\ \alpha_7 &= \varepsilon_9 - \varepsilon_{10} \;=\; (0,\ldots,0,1,-1)\,, \\ \alpha_8 &= \varepsilon_8 + \varepsilon_9 + \varepsilon_{10} \;=\; (0,\ldots,0,1,1,1)\,. \end{aligned} \tag{3.4}$$

The advantage of such a basis is to give a rank-, and hence dimension-, independent formulation of Π, which is not the case for an orthogonal basis e_i. Preserving the scalar product on the root system requires the physical basis to be endowed with the following scalar product:

$$(\alpha|\beta) = \sum_{i=11-r}^{10} n^i m^i - \frac{1}{9} \sum_{i,j=11-r}^{10} n^i m^j \tag{3.5}$$

for $\alpha = \sum_{i=11-r}^{10} n^i \varepsilon_i$ and $\beta = \sum_{i=11-r}^{10} m^i \varepsilon_i$ (note that the basis elements satisfy $(\varepsilon_i|\varepsilon_j) = \delta_{ij} - (1/9)$ and have length $2\sqrt{2}/3$).

In fact, writing this change of basis as $\alpha_i = (R^{-1})_i^{\ j} \varepsilon_j$, we can invert this relation (which leads to the matrix R given in appendix A.I)) and obtain the metric corresponding to the scalar product (3.5), given in terms of the Cartan matrix as:

$$g_\varepsilon = RAR^\top\,, \tag{3.6}$$

in the simply-laced case we are interested in.

As seen in Section 2.2, the Cartan matrix of the $\mathfrak{e}_r$ serie is a realization of $(\mathfrak{h}, \Pi, \Pi^\vee)$, where now $\Pi^\vee = \{\alpha^\vee_{9-r}, \ldots, \alpha^\vee_8\} \cong \Pi$. Then, from $A_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle = (\alpha_i|\alpha_j)$ we have:

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots \quad \ddots & \vdots & \vdots \\ 0 \cdots & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 \cdots & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 \cdots & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 \cdots & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 \cdots & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix} \tag{3.7}$$

and A corresponds to a Dynkin diagram of the type 1.

The simple coroots, in turn, form a basis of the derived Cartan subalgebra $\mathfrak{h}'$, and we may choose (or alternatively define) $H_i \doteq \alpha_i^\vee$, $\forall i = 9-r, \ldots, 8$. Since we consider

[3] The physical basis makes sense only in cases where there are scalars coming from compactification of the 3-form, which excludes the first two algebras of the serie.

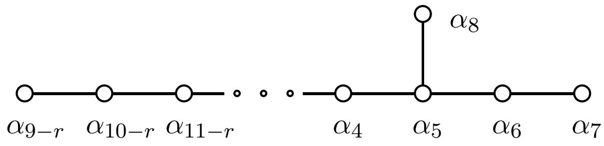

Figure 1: Dynkin diagram of the E_r serie

simply laced-cases only, the relation $\alpha_i^\vee = (AR^\top)_{ij}\varepsilon^{\vee j}$ determines the dual physical basis for $r \neq 9$, *i.e.* $D \neq 2$:

$$\begin{aligned} H_{9-r} &= \varepsilon^{\vee\, 11-r} - \varepsilon^{\vee\, 12-r} = (1,-1,0,\dots,0) \\ &\vdots \\ H_7 &= \varepsilon^{\vee\, 9} - \varepsilon^{\vee\, 10} = (0,\dots,0,1,-1)\,, \\ H_8 &= -\frac{1}{3}\sum_{i=11-r}^{7} \varepsilon^{\vee\, i} + \frac{2}{3}\sum_{i=8}^{10} \varepsilon^{\vee\, i} = \frac{1}{3}(-1,\dots,-1,2,2,2)\,. \end{aligned} \tag{3.8}$$

In the same spirit as before, this dual basis is equipped with a scalar product given in terms of the metric $g_\varepsilon^\vee = (g_\varepsilon)^{-1}$ as:

$$B(H,H') = \sum_{i=11-r}^{10} h_i h'_i + \frac{1}{9-r}\sum_{i,j=11-r}^{10} h_i h'_j\,, \qquad \text{for } r \neq 9 \tag{3.9}$$

for two elements $H = \sum_{i=11-r}^{10} h_i \varepsilon^{\vee\, i}$ and $H' = \sum_{i=11-r}^{10} h'_i \varepsilon^{\vee\, i}$. In the affine case $r = 9$, the Cartan matrix is degenerate. In order to determine $B(H,H')$, one has to work in the whole Cartan subalgebra $\mathfrak{h}$, and not only in the derived one, and include a basis element related to the scaling operator d. Consequently, there is no meaningful physical basis in this case.

Not surprisingly, we recognize in (3.9) the Killing form of $\mathfrak{e}_r$ restricted to $\mathfrak{h}(\mathfrak{e}_r)$. Since the dual metric is the inverse of $g_\varepsilon^\vee$, the duality bracket is defined as usual as $\langle \varepsilon_i, \varepsilon^{\vee\, j}\rangle = \varepsilon_i(\varepsilon^{\vee\, j}) = \delta_i^{\ j}$, so that consistently:

$$\langle \alpha^\vee, \beta\rangle = (\nu(\alpha^\vee)|\beta) = \beta(H_\alpha)\,. \tag{3.10}$$

Since $\mathfrak{e}_r$ is simply-laced, $\nu(\alpha^\vee) = \alpha$, and we then have various ways of expressing the Cartan matrix:

$$A_{ij} \doteq \alpha_i(H_j) \equiv (\alpha_i|\alpha_j) \equiv B(H_i,H_j)\,. \tag{3.11}$$

3.1.1 A choice for structure constants

We now fix the conventions for the E_r serie that will hold throughout the paper. For obvious reasons of economy, we introduce the following compact notation to characterize $\mathfrak{e}_r$ generators:

Notation 3.2 Let X_α be a generator of the root space $(\mathfrak{e}_r)_\alpha$, or of the dual subspace $\mathfrak{h}_\alpha \subset \mathfrak{h}$ for some root $\alpha = \sum_{i=9-r}^{8} k^i \alpha_i \in \Delta(\mathfrak{e}_r)$. We write the corresponding generator as

$$X_{(9-r)^{k^{9-r}}\dots 8^{k^8}} \quad \text{instead of} \quad X_{k^{9-r}\alpha_{9-r}+\dots+k^8\alpha_8}\,.$$

For example we will write:

$$E_{45^2678} \quad \text{instead of} \quad E_{\alpha_4+2\alpha_5+\alpha_6+\alpha_7+\alpha_8}\,,$$

and similarly for F and H. Sometimes, we will also write $\alpha_{(9-r)^{k^{9-r}}\dots 8^{k^8}}$ instead of $\sum_{i=9-r}^{8} k^i\alpha_i$.

Furthermore, δ always refers to the isotropic root of $\mathfrak{e}_9$, namely $\delta = \delta_{E_9} = \alpha_{0 1^2 2^3 3^4 4^5 5^6 6^4 7^2 8^3}$, $c = H_\delta$ to its center, and d to its usual derivation operator $d = d_{E_9}$. Possible subscripts added to δ, c and d will be used to discriminate $\mathfrak{e}_9$ objects from objects belonging to its subalgebras.

Moreover, we use for $\mathfrak{e}_9$ the usual construction based on the loop algebra $\mathcal{L}(\mathfrak{e}_8) \doteq \mathbb{C}[z,z^{-1}]\otimes\mathfrak{e}_8$, $z\in\mathbb{C}$. The affine KMA $\mathfrak{e}_9$ is then obtained as a central extension thereof:

$$\mathfrak{e}_9 = \mathcal{L}(\mathfrak{e}_8)\oplus\mathbb{C}c\oplus\mathbb{C}d\,.$$

and is spanned by the basis of vertex operators satisfying

$$\begin{aligned}
[z^m\otimes H_i, z^n\otimes H_j] &= m\delta_{ij}\delta_{m+n,0}\,c\,,\\
[z^m\otimes H_i, z^n\otimes E_\alpha] &= \langle\alpha_i^\vee,\alpha\rangle z^{m+n}\otimes E_\alpha\,,\\
[z^m\otimes E_\alpha, z^n\otimes E_\beta] &= \begin{cases} \mathcal{N}_{\alpha,\beta} z^{m+n}\otimes E_{\alpha+\beta}\,, & \text{if } \alpha+\beta\in\Delta(\mathfrak{e}_9)\\ z^{m+n}\otimes H_\alpha + m\delta_{m+n,0}\,c\,, & \text{if } \alpha=-\beta\\ 0\,, & \text{otherwise}\end{cases}\,.
\end{aligned} \tag{3.12}$$

In addition, the Hermitian scaling operators $d = z\frac{d}{dz}$ defined from $z\in S^1$ normalizes $\mathcal{L}(\mathfrak{e}_8)$: $[d, z^n\otimes X] = nz^n\otimes X$, $\forall X\in\mathfrak{e}_8$.

In $\mathfrak{e}_{10}$, for which there is no known vertex operator construction yet, we rewrite the $\mathfrak{e}_9$ subalgebra according to the usual prescriptions for KMAs by setting: $d = -H_{-1}$, $E^a_{n\delta} = z^n\otimes H_a$, with $a = 1,..,8$ its multiplicity, $E_{\alpha+n\delta} = z^n\otimes E_\alpha$ and $E_{-\alpha+n\delta} = z^n\otimes F_\alpha$, and similarly for negative-root generators.

Finally, there is a large number of mathematically acceptable sign conventions for the structure constants $\mathcal{N}_{\alpha,\beta}$, as long as one satisfies the anti-commutativity and Jacobi identity of the Lie bracket, as explained in Definition 2.5. If one decides to map physical fields to generators of a KMA, which will eventually be done in this paper, one has to make sure that the adjoint action of a rotation with positive angle leads to a positive rotation of all physical tensors carrying a covariant index affected by it. This physical requirement imposes more stringent constraints on the structure constants. Though perhaps not the most natural choice from a mathematical point of view, we fix signs according to a lexicographical ordering for level 0 ($\mathfrak{sl}(r,\mathbb{R})-$) roots, but according to an ordering based on their height for roots of higher level in α_8. More concretely, if $\alpha = \alpha_{j\dots k}$ has level 0, we set:

$$\mathcal{N}_{\alpha_i,\alpha} = \begin{cases} 1 \text{ if } i<j\\ -1 \text{ if } k<i\end{cases}\,.$$

On the other hand, we fix $\mathcal{N}_{5,8} = +1$, and always take the positive sign when we lengthen a chain of simple roots of level $l > 0$ by acting with a positive simple root generator from the left, i.e.:

$$\mathcal{N}_{\alpha_i,\alpha} = 1\,,\ \forall\alpha \text{ s.t. } l(\alpha) > 0\,.$$

Structure constants for two non-simple and/or negative roots are then automatically fixed by these choices.

3.2 Toroidally reduced $11D$ supergravity: scalar fields and roots of E_{11-D}

In this section, we rephrase the mapping between scalar fields of 11D supergravity on T^q, $q \geqslant 3$, and the roots of its finite U-duality algebras, in a way that will make clear the extension to the infinite-dimensional case. We start with $\mathcal{N} = 1$ classical $11D$ supergravity, whose bosonic sector is described by the Lagrangian:

$$S_{11} = \frac{1}{l_P^9}\int d^{11}x\, e\Big(R - \frac{l_P^6}{2\cdot 4!}(G_4)^2\Big) + \frac{1}{2\cdot 3!}\int C_3\wedge G_4\wedge G_4 \tag{3.13}$$

where the four-form field strength is exact: $G_4 = dC_3$. There are various conventions for the coefficients of the three terms in the Lagrangian (3.13), which depend on how one defines the fermionic sector of the theory. In any case, the factors of Planck length can be fixed by dimensional analysis. Here, we adopt the conventions of [23], where we have, in units of length

$$[g_{AB}] = 2\,,\quad [C_{ABC}] = 0\,,\quad [d] = [dx] = 0\,.$$

As a consequence of the above: $[R] = -2$.

The action (3.13) rescales homogeneously under:

$$g_{AB} \to M_P^2 g_{AB}\,,\quad C_{ABC} \to M_P^3 C_{ABC} \tag{3.14}$$

which eliminates all l_P terms from the Einstein-Hilbert and gauge Lagrangian while rescaling the Chern-Simons (CS) part by l_P^{-9}, and renders, in turn, g_{AB} dimensionless. This is the convention we will adopt in the following which will fix the mapping between the E_{11-D} root system and the fields parametrizing the scalar manifold of the reduced theory for $D \geqslant 3$. How to extend this analysis to the conjectured affine and hyperbolic U-duality groups E_9 and E_{10} will be treated in the next section.

The Kaluza-Klein reduction of the theory to $D \geqslant 3$ dimensions is performed according to the prescription

$$ds_{11}^2 = e^{\frac{\sqrt{2}}{D-2}(\rho_D|\varphi)}ds_D^2 + \sum_{i=D}^{10} e^{-\sqrt{2}(\varepsilon_i|\varphi)}\left(\tilde{\gamma}^i{}_j dx^j + \mathcal{A}_1^i\right)^2\,, \tag{3.15}$$

with $\tilde{\gamma}^i{}_j = (\delta^i{}_j + \mathcal{A}^i_{0\,j})$ with $i < j$ for $\mathcal{A}^i_{0\,j}$. The compactification vectors ε_i are the ones defining the physical basis of Definition 3.1, and can be expressed in the orthonormal basis $\{e_i\}_{i=1}^{11-D}$, $e_i\cdot e_j = \delta_{ij}$, as

$$\varepsilon_k = -\sum_{i=1}^{10-k}\frac{1}{\sqrt{(10-i)(9-i)}}e_i + \sqrt{\frac{k-2}{k-1}}e_{11-k}\,.\quad \text{for } k \leqslant 8\,.$$

In the $D = 2, 1$ cases, the additional vectors completing the physical basis are defined formally, without reference to the compactification procedure.

Accordingly, the vector of dilatonic scalars can be expanded as $\varphi = \sum_{i=1}^{11-D} \varphi_i e_i$. We will however choose to stick to the physical basis. The expression of ε_k in terms of the orthonormal basis will help to connect back to the prescription of [6] and [57, 58]. In this respect, the scalar product $(\,|\,)$ used in expression (3.15) is precisely the product on the root system (3.5). Finally, we also introduce the "threshold" vector

$$\rho_D = \sum_{i=D}^{10} \varepsilon_i \tag{3.16}$$

which will be crucial later on when studying the structure of Minkowskian objects in E_{10}.

From expression (3.15), we see that the elfbein produces $(11-D)$ one-forms $\mathcal{A}^i_1$ and $(11-D)(12-D)/2$ scalars $\mathcal{A}^i_{0\,j}$, whereas the three form generates the following two-, one- and zero-form potentials: $(11-D)$ $C_{2\,i}$, $(11-D)(10-D)/2$ $C_{1\,ij}$ and $(11-D)(10-D)(9-D)/6$ $C_{0\,ijk}$. The reduction of the $11D$ action (3.13) to any dimension greater than two reads:

$$\begin{aligned} e^{-1}\mathcal{L} = {} & R - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2\cdot 4!} e^{\sqrt{2}(\kappa|\varphi)} (\underline{G_4})^2 - \frac{1}{2\cdot 3!} \sum_i e^{\sqrt{2}(\kappa_i|\varphi)} (\underline{G_{3\,i}})^2 \\ & - \frac{1}{2\cdot 2!} \sum_{i<j} e^{\sqrt{2}(\kappa_{ij}|\varphi)} (\underline{G_{2\,ij}})^2 - \frac{1}{2\cdot 2!} \sum_i e^{\sqrt{2}(\lambda_i|\varphi)} (\underline{\mathcal{F}^i_2})^2 \\ & - \frac{1}{2} \sum_{i<j<k} e^{\sqrt{2}(\kappa_{ijk}|\varphi)} (\underline{G_{1\,ijk}})^2 - \frac{1}{2} \sum_{i<j} e^{\sqrt{2}(\lambda_{ij}|\varphi)} (\underline{\mathcal{F}^i_{1\,j}})^2 + e^{-1}\mathcal{L}_{CS} \end{aligned} \tag{3.17}$$

where $\mathcal{L}_{CS}$ is the reduction of the CS-term $C_3 \wedge G_4 \wedge G_4$, and again indices run according to $i, j, k = D, .., 10$. The field strengths appearing in the above kinetic term exhibit the exterior derivative of the corresponding potentials as leading term, but contain additional non-linear Kaluza-Klein modifications. For instance:

$$\begin{aligned} & \underline{G_4} = G_4 - \gamma^i{}_j G_{3\,i} \wedge \mathcal{A}^j_1 + \gamma^i{}_k \gamma^j{}_l G_{2\,ij} \wedge \mathcal{A}^k_1 \wedge \mathcal{A}^l_1 + \dots \,, && \underline{\mathcal{F}^i_2} = \mathcal{F}^i_2 - \gamma^j{}_k \mathcal{F}^i_{1\,j} \wedge \mathcal{A}^k_1 \,, \\ & \vdots && \underline{\mathcal{F}^i_{1\,j}} = \gamma^k{}_j \mathcal{F}^i_{1\,k} \,. \\ & \underline{G_{2\,ij}} = \gamma^m_i \gamma^n_j G_{2\,mn} - \gamma^l{}_i \gamma^m_j \gamma^n{}_k G_{1\,lmn} \wedge \mathcal{A}^k_1 \,, && \\ & \underline{G_{1\,ijk}} = \gamma^l{}_i \gamma^m_j \gamma^n{}_k G_{1\,lmn} \,, && \end{aligned} \tag{3.18}$$

with $\gamma^i{}_j = (\tilde{\gamma}^{-1})^i{}_j$, the not-underlined field strengths being total derivatives: $G_{(n)\,i_1\cdots i_l} = dC_{(n-1)\,i_1\cdots i_l}$ and $\mathcal{F}^i_{(n)\,i_1\cdots i_l} = d\mathcal{A}^i_{(n-1)\,i_1\cdots i_l}$, where n is the rank of the form. The whole set of field strengths and the details of the reduction of the CS term are well known and can be found in [59].

The global symmetry of the scalar manifold which, upon quantization, is conjectured to become the discrete U-duality symmetry of the theory is encoded in the compactification vectors $\overline{\Delta} = \{\kappa; \kappa_i; \kappa_{ij}; \kappa_{ijk}; \lambda_i; \lambda_{ij}\}$ appearing in the Lagrangian (3.17). As pointed out previously, this global symmetry will only be manifest if potentials of rank $D-2$ are dualized to scalars, thereby allowing gauge symmetries to be replaced by internal ones. In each dimension D, this will select a subset of $\overline{\Delta}$ to form the positive root system of E_{11-D}. One has to keep in mind, however, that in even space-time dimensions, this rigid symmetry is usually only realized on the field strengths themselves, and not on the potentials. This is attributable to the customary difficulty of writing a covariant lagrangian for self-dual fields [4]. In even dimensional cases then, the E_{11-D} symmetry appears as a local field transformation on the solution of the equations of motion.

We now give the whole set $\overline{\Delta}$ in the physical basis. One has to bear in mind that some of these vectors become roots only in particular dimensions, and thus do not, in general, have squared length equal to 2. In constrast, λ_{ij} and κ_{ijk} are always symmetries of the scalar manifold, and can therefore be directly translated into positive roots of E_{11-D} for the first two levels $l = 0, 1$ in α_8. We have for $i < j < k$:

$$
\begin{aligned}
l = 0 :&\ W_{\mathrm{KKp}} \ni \lambda_{ij} = \varepsilon_i - \varepsilon_j \\
l = 1 :&\ \ W_{\mathrm{M2}} \ni \kappa_{ijk} = \varepsilon_i + \varepsilon_j + \varepsilon_k
\end{aligned}
$$

In fact, they build orbits of E_{11-D} under the Weyl group of $SL(11 - D, \mathbb{R})$, which we denote by W_{KKp} and W_{M2}, anticipating results from M-theory on T^{10} which associates λ_{ij} with Euclidean KK particles and κ_{ijk} with Euclidean M2-branes. So, in $10 \geqslant D \geqslant 6$, since no dualization occurs, the root system of the U-duality algebra is completely covered by W_{KKp} and W_{M2}, with the well known results $G^U = \{SO(1,1); GL(2,\mathbb{R}); SL(3,\mathbb{R}) \times SL(2,\mathbb{R}); SL(5,\mathbb{R}); SO(5,5)\}$, for $D = \{10; 9; 8; 7; 6\}$.

For $D = 5$, we dualize $\underline{G}_4 = e^{-\kappa\cdot\varphi} * \underline{\widetilde{G}}_1$, with $-\kappa = \theta_{E_6} = ((1)^6)$ the highest root of E_6, which constitutes a Weyl orbit all by itself. For highest roots of the Lie algebra relevant to our purpose, we refer the reader to Appendix A ii). In $D = 4$, we dualize $(\underline{G}_{3\,i}) = e^{-\kappa_i\cdot\varphi} * \underline{\widetilde{G}}_{1\,i}$, with $-\kappa_i = ((1)^{i-1}, 0_i, (1)^{7-i})$ forming the Weyl orbit of θ_{E_7} (which contains θ_{E_6}). Finally, in $D = 3$, dualizing $\underline{G}_{2\,ij} = e^{-\kappa_{ij}\cdot\varphi} * \underline{\widetilde{G}}_{1\,ij}$ and $\underline{\mathcal{F}}^i_2 = e^{-\lambda_i\cdot\varphi} * \underline{\widetilde{\mathcal{F}}}^i_1$ increases the size of the former θ_{E_7} Weyl orbit and creates the remaining θ_{E_8} orbit:

$$
\begin{aligned}
D = 5\ l = 2 :&\ \ W_{\mathrm{M5}} &&\ni -\kappa = \tfrac{3}{D-2}\rho_D \\
D = 4\ l = 2 :&\ \ W_{\mathrm{M5}} &&\ni -\kappa_i = \tfrac{2}{D-2}\rho_D - \varepsilon_i \\
D = 3\ l = 2 :&\ \ W_{\mathrm{M5}} &&\ni -\kappa_{ij} = \tfrac{1}{D-2}\rho_D - \varepsilon_i - \varepsilon_j \\
l = 3 :&\ W_{\mathrm{KK7M}} &&\ni -\lambda_i = \tfrac{1}{D-2}\rho_D + \varepsilon_i
\end{aligned}
\tag{3.19}
$$

For the same reason as before, we denote these two additional orbits W_{M5} and W_{KK7M} since they will be shown to describe totally wrapped Euclidean M5-branes and KK monopoles. For $D = 3$, for instance, we can check that $\dim W_{\mathrm{KKp}} = \dim W_{\mathrm{M5}} = 28$, $\dim W_{\mathrm{M2}} = 56$ and $\dim W_{\mathrm{KK7M}} = 8$, which reproduces the respective number of scalars coming from the KK gauge fields, 3-form, and their magnetic duals, and verifies $\dim \Delta_+(E_8) = \sum_i \dim W_i$.

For what follows, it will turn out useful to take advantage of the dimensionless character of the vielbein, resulting from the rescaling (3.14), to rewrite the internal metric in terms

[4]In some cases -for the $11D$ five-brane for instance-, this can be achieved by resorting to the Pasti-Sorokin-Tonin formalism.

of the duality bracket (3.10):

$$ds^2_{11-D} = \sum_{i=D}^{10} e^{2\langle H_R,\varepsilon_i\rangle}\delta_{ij}\tilde{\gamma}^i{}_k\tilde{\gamma}^j{}_l dx^k \otimes dx^l \tag{3.20}$$

with $H_R = \sum_{i=D}^{10} \ln(M_P R_i)\varepsilon^{\vee i}$. Thus in particular: $e^{-\sqrt{2}(\varepsilon_i|\varphi)} = (M_P R_i)^2$. In this convention, the scalar Lagrangian for $D = 3$ reads

$$\begin{aligned}&-eg^{AB}(g_\varepsilon^\vee)^{ij}\left(\frac{\partial_A R_i}{R_i}\right)\left(\frac{\partial_B R_j}{R_j}\right) - \frac{1}{2}e\sum_{i<j<k}\frac{1}{(M_P^3 R_i R_j R_k)^2}(\underline{G}_{1\,ijk})^2\\&-\frac{1}{2}e\sum_{i<j}\left(\frac{R_j}{R_i}\right)^2(\underline{\mathcal{F}}^i_{1\,j})^2 - \frac{1}{4}e\sum_{i<j}\left(\frac{R_iR_j}{M_P^6 V_8}\right)^2(\underline{\widetilde{G}}_{1\,ij})^2 - \frac{1}{4}e\sum_i\left(\frac{1}{M_P^9 R_i V_8}\right)^2(\underline{\widetilde{\mathcal{F}}}^i_1)^2\end{aligned} \tag{3.21}$$

with the dual metric $(g_\varepsilon^\vee)^{ij} = \delta^{ij} + (D-2)^{-1}\sum_{k,l}\delta^{ik}\delta^{lj}$ (3.9) and the internal volume $V_8 = \prod_{i=3}^{10} R_i$. Clearly, the coefficients[5] in front of the one-form kinetic terms reproduce the inverse squared tensions for totally wrapped Euclidean KK particles, M2-branes, M5-branes and KK monopoles (KK7M-branes). This will be the touchstone of our analysis, and in the next section we will present, following [21], how the corresponding Minkowskian branes arise in $D = 1$ in the framework of $E_{10|10}/K(E_{10|10})$ hyperbolic billiards.

3.3 Non-linear realization of supergravity: the triangular and symmetric gauges

The final step towards unfolding the hidden symmetry of the scalar manifold of the reduced theory consists in showing that one can construct its Langrangian density as a coset σ-model from a non-linear realization. Here, we rederive, in the formalism we use later on, only the most symmetric $D = 3$ case, since the $D \geqslant 4$ constructions are obtainable as restriction thereof by referring to table (3.19). For the detailed study of the $D > 3$ cases, see [6].

Furthemore, the use of a parametrization of the coset sigma-model based on the Borel subalgebra of the U-duality algebra, called triangular gauge, is crucial to this type of non-linear realization. In contrast, we will show in the second part of this section, that the most natural setup to treat orbifolds of the corresponding supergravities is given by a parametrization of the coset based on the Cartan decomposition of the U-duality algebra. We refer to this choice as the symmetric gauge.

In $D = 3$, the non-linear realization of the scalar manifold is based on the group element:

$$\begin{aligned}g = \exp\Big[-\frac{1}{\sqrt{2}}\sum_i \ln(M_P R_i)\varepsilon^{\vee i}\Big]\cdot\Big(\prod_{i<j} e^{\mathcal{A}^i{}_j K^{+j}_i}\Big)\cdot\exp\Big[\sum_{i<j<k} C_{ijk}Z^{+ijk}\Big]\cdot\\ \cdot\exp\Big[\sum_{i_1<..<i_6}\widetilde{C}_{i_1..i_6}\widetilde{Z}^{+i_1..i_6}\Big]\cdot\exp\Big[\sum_{i_1<..<i_8,j}\widetilde{\mathcal{A}}^j{}_{i_1..i_8}\widetilde{K}^{+i_1..i_8}_j\Big],\end{aligned} \tag{3.22}$$

[5] Note that the interpretation of these coefficients is somewhat different than in [23], since here we are working in the conformal Einstein frame.

where the dual potentials are reformulated to exhibit a tensorial rank that would generalize to $D < 3$ (we drop all 0 subscripts since we are only dealing with scalars). In particular, they are related to the $D = 3$ dual potentials as $\widetilde{C}_{i_1..i_6} = \frac{1}{2!}\epsilon_{i_1..i_6 kl}\widetilde{C}_0{}^{kl}$ and $\widetilde{\mathcal{A}}^j{}_{i_1..i_8} = \epsilon_{i_1.i_8}\widetilde{\mathcal{A}}^j_0$, by the totally antisymmetric rank 8 tensor of $SL(8,\mathbb{R})$, $\epsilon_{i_1..i_8}$. These entered expression (3.21) as $\widetilde{G}_{1\,ij} = d\widetilde{C}_{0\,ij}$ and $\widetilde{\mathcal{F}}^i_1 = d\widetilde{\mathcal{A}}^i_0$.

The group element (3.22) is built out of the Borel subalgebra of E_8, which is spanned by the following raising operators

$$\begin{aligned}
[K_i^{+\,j}, K_k^{+\,l}] &= \delta^j_k K_i^{+\,l} - \delta^l_i K_k^{+\,j}\,, \\
[K_i^{+\,j}, Z^{+\,k_1k_2k_3}] &= -3\delta_i^{[k_1} Z^{+\,|j|k_2k_3]}\,, \qquad [K_i^{+\,j}, \widetilde{Z}^{+\,k_1..k_6}] = -6\delta_i^{[k_1}\widetilde{Z}^{+\,|j|k_2..k_6]}\,, \\
[K_i^{+\,j}, \widetilde{K}_k^{+\,k_1..k_8}] &= \delta^j_k \widetilde{K}_i^{+\,k_1..k_8}\,, \\
[Z^{+\,i_1i_2i_3}, Z^{+\,i_4i_5i_6}] &= -\widetilde{Z}^{+\,i_1..i_6}\,, \qquad [Z^{+\,i_1i_2i_3}, \widetilde{Z}^{+\,i_4..i_9}] = -3\widetilde{K}^{+\,[i_1|i_2i_3]i_4..i_9}\,,
\end{aligned} \tag{3.23}$$

and the Cartan subalgebra, acting on the former as (without implicit summations on repeated indices)

$$\begin{aligned}
[\varepsilon^{\vee i}, K_j^{+\,k}] &= \lambda_{jk}(\varepsilon^{\vee i})\, K_j^{+\,k}\,, \qquad [\varepsilon^{\vee i}, Z^+_{jkl}] = \kappa_{jkl}(\varepsilon^{\vee i})\, Z^+_{jkl}\,, \\
[\varepsilon^{\vee i}, \widetilde{Z}^+_{j_1..j_6}] &= -\sum_{l<m} \epsilon_{j_1..j_6}{}^{lm}\,\kappa_{lm}(\varepsilon^{\vee i})\widetilde{Z}^+_{j_1..j_6}\,, \quad [\varepsilon^{\vee i}, \widetilde{K}_j^{+\,k_1..k_8}] = -\kappa_j(\varepsilon^{\vee i})\,\widetilde{K}_j^{+\,k_1..k_8}\,.
\end{aligned}$$

Anticipating the extension to $D = 2, 1$, we redefine positive roots $\kappa_{i_1..i_6} \doteq \sum_{l<m}\epsilon_{i_1..i_6}{}^{lm}\kappa_{lm}$ and $\lambda_{j|i_1..i_8} \doteq \epsilon_{i_1..i_8}\kappa_j$ corresponding to the generators $\widetilde{Z}$ and $\widetilde{K}$. The scalar Lagrangian in D dimensions is then expressible as a coset sigma-model, obtained from the algebraic field strength $\mathcal{G} \doteq g^{-1}dg$

$$\begin{aligned}
\mathcal{G} = &-\frac{1}{\sqrt{2}}\sum_i \ln(M_P R_i)\varepsilon^{\vee i} + \sum_{i<j<k} e^{\langle H_R,\kappa_{ijk}\rangle}\underline{G}_{1\,ijk}Z^{+\,ijk} + \sum_{i<j} e^{\langle H_R,\lambda_{ij}\rangle}\underline{\mathcal{F}}^i_{1\,j}K_i^{+\,j} \\
&+\sum_{i_1<..<i_6} e^{-\langle H_R,\kappa_{i_1..i_6}\rangle}\underline{\widetilde{G}}_{1\,i_1..i_6}\widetilde{Z}^{+\,i_1..i_6} + \sum_{i_1<..<i_8,j} e^{-\langle H_R,\lambda_{j|i_1..i_8}\rangle}\underline{\widetilde{\mathcal{F}}}_1^{\,j}{}_{i_1..i_8}\widetilde{K}_j^{+\,i_1..i_8}\,.
\end{aligned} \tag{3.24}$$

This particular parametrization of the coset $\mathfrak{e}_{8|8}/\mathfrak{k}(\mathfrak{e}_{8|8})$ is known as the Iwasawa decomposition. In other words, the split form $\mathfrak{g}^U = \mathfrak{e}_{11-D|11-D}$ decomposes as a sum of closed factors $\mathfrak{g}^U = \mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$, where $\mathfrak{k}$ is its maximal compact subalgebra. Then, the coset $\mathfrak{g}^U/\mathfrak{k} = \mathfrak{a}\oplus\mathfrak{n}$ is parametrized by the direct sum of an abelian and a nilpotent subalgebra. This can be interpreted as a "gauge" choice, where the coset elements are either diagonal (Cartan generators) or upper triangular (Borel, or positive root, generators). In the following, this choice will be referred to as the *triangular gauge*.

The negative root generators can be retrieved from the Borel subalgebra by defining the appropriate transposition operation. Since we want it to be applicable to $\mathfrak{g}^U = \mathfrak{e}_{11-D|11-D}$ $\forall D$, and not only to U-duality algebras with orthogonal maximal compact subalgebra, we construct it as in [6] out of the Cartan involution ϑ as $T(X) = -\vartheta(X)$, $\forall X \in \mathfrak{g}^U$. This induces a corresponding generalized transposition [6, 7] on the group level denoted by: $T(\mathcal{X}) = \Theta(\mathcal{X}^{-1})$, $\forall \mathcal{X} \in G^U$. In the present case, since $\mathfrak{g}^U$ is the split form, we have $\vartheta = \vartheta_C$, the latter being the Chevalley involution.

By requiring the following normalizations:

$$\mathrm{Tr}(\varepsilon^{\vee i}\varepsilon^{\vee j}) = 2(g_\varepsilon^\vee)^{ij}\,, \qquad \mathrm{Tr}\Big(K_i{}^j T(K_k{}^l)\Big) = \delta_{ik}\delta^{jl}\,,$$
$$\mathrm{Tr}\Big(X^{i_1..i_p|j_1..j_q} T(X^{k_1..k_p|l_1..l_q})\Big) = p!q!\delta^{[i_1}_{k_1}\cdots\delta^{i_p]}_{k_p}\delta^{[j_1}_{l_1}\cdots\delta^{j_q]}_{l_q}\,,$$

where $X^{i_1..i_p|j_1..j_q}$ stand for the remaining generators of (3.23), the bosonic scalar Lagrangian (3.21) is readily obtained from the coset sigma-model

$$\mathcal{L} = -\frac{1}{2}e\mathrm{Tr}\left[g^{-1}\partial g(\mathbb{1}+T)g^{-1}\partial g\right] \equiv \frac{1}{4}e\mathrm{Tr}(\partial\mathcal{M}^{-1}\partial\mathcal{M}) \tag{3.25}$$

where $\mathcal{M} = gT(g)$ is the internal σ-model metric. The equations of motion for the moduli of the theory are then summarized in the Maurer-Cartan equation: $d\mathcal{G} = \mathcal{G}\wedge\mathcal{G}$. By adding the negative root generators, we restore the $K(E_{11-D})$ local gauge invariance, and enhance the coset to the full continuous U-duality group. In $D=3$, for instance, we thus recover the dimension of E_8 as:

$$\begin{aligned}248 \,=\, & 8(H_i) + 28(K_i^{+j}) + 56(Z^{+ijk}) + 28(\widetilde{Z}^{+i_1..i_6}) + 8(\widetilde{K}_j^{+i_1..i_8}) \\ & +\overline{28}\left(T(K_i^{+j})\right) + \overline{56}\left(T(Z^{+ijk})\right) + \overline{28}\left(T(\widetilde{Z}^{+i_1..i_6})\right) + \overline{8}\left(T(\widetilde{K}_j^{+i_1..i_8})\right)\end{aligned}$$

Note that the triangular gauge is not preserved by a rigid left transformation U from the symmetry group G^U: $g(x) \to Ug(x)$ for $g \in G^U/K(G^U)$. This leaves $\mathcal{G}$ invariant but will generally send g out of the positive root gauge. We will then usually need a local compensator $h(x) \in K(G^U)$ to bring it back to the original gauge. So the Lagrangian (3.25) is kept invariant by the compensated transformation $g(x) \to Ug(x)h(x)^{-1}$ which sends: $\mathcal{M} \to U\mathcal{M}T(U)$, provided $hT(h) = \mathbb{1}$.

If the triangular gauge is the natural choice to obtain a closed non-linear realization of a coset sigma-model, it will show to be quite unhandy when trying to treat orbifolds of reduced $11D$ supergravity and M-theory. In this case, a parametrization of the coset based on the Cartan decomposition into eigenspace of the Chevalley involution $\mathfrak{g}^U = \mathfrak{k}\oplus\mathfrak{p}$ is more appropriate. In other words, one starts from an algebraic field strength valued in $\mathfrak{g}^U$:

$$\widetilde{g}^{-1}d\widetilde{g} = \mathcal{P} + \mathcal{Q} \quad , \tag{3.26}$$

so that the coset is parametrized by:

$$\mathcal{P} = \frac{1}{2}(\mathbb{1}+T)\,g^{-1}dg\,. \tag{3.27}$$

and $\mathcal{Q}$ ensures local (now unbroken) $K(G^U)$ invariance of the model. Note that the Lagrangian (3.25) is, as expected, insensitive to this different parametrization since $\frac{1}{4}e\mathrm{Tr}$ $(\partial\mathcal{M}^{-1}\partial\mathcal{M}) \equiv -e\mathrm{Tr}\left[\mathcal{P}T(\mathcal{P})\right]$.

We then associate symmetry generators to the moduli of compactified $11D$ supergravity / M-theory in the following fashion: for economy, we will denote all the Borel generators of $\mathfrak{sl}(11-D,\mathbb{R}) \subset \mathfrak{g}^U$ by $K_i^{+\,j}$ for $i \leqslant j$, by setting in particular $K_i^{\pm\,i} = K_i^{\ i} \doteq \varepsilon^{\vee\, i}$. Using relation (3.8), the Cartan generators can now be reexpressed as

$$H_i = K_{i+2}^{\ \ i+2} - K_{i+3}^{\ \ i+3}\,, \quad i = 1,..,7\,, \qquad H_8 = -\frac{1}{3}\sum_{i=1}^{5} K_{i+2}^{\ \ i+2} + \frac{2}{3}\left(K_8^{\ 8} + K_9^{\ 9} + K_{10}^{\ \ 10}\right)\,.$$

The dictionary relating physical moduli and coset generators can then be established for all moduli fields corresponding to real roots of level $l = 0, 1, 2, 3$, generalizing to $D = 2, 1$ the previous result (3.19). We will denote the generalized transpose of a Borel generator X^+ as $X^- \doteq T(X^+)$.

modulus	generator	physical basis
$\ln(M_P R_i)$	$K_i^{\ i}$	$\varepsilon^{\vee\, i}$
$\mathcal{A}^i_{\ j}$	$K_i^{\ j} = \frac{1}{2}\left(K_i^{+\,j} + K_i^{-\,j}\right)$	$\varepsilon_i - \varepsilon_j$
C_{ijk}	$Z^{ijk} = \frac{1}{2}\left(Z^{+\,ijk} + Z^{-\,ijk}\right)$	$\varepsilon^i + \varepsilon^j + \varepsilon^k$
$\widetilde{C}_{i_1..i_6}$	$Z^{i_1..i_6} = \frac{1}{2}\left(Z^{+\,i_1..i_6} + Z^{-\,i_1..i_6}\right)$	$\sum_{l=1}^{6} \varepsilon_{i_l}$
$\widetilde{\mathcal{A}}^j_{\ i_1..i_8},\ j \in \{i_1,..,i_8\}$	$\widetilde{K}_j^{\ i_1..i_8} = \frac{1}{2}\left(\widetilde{K}_j^{+\,i_1..i_8} + \widetilde{K}_j^{-\,i_1..i_8}\right)$	$\sum_{l=1}^{8}(1+\delta_j^{i_l})\varepsilon_{i_l}$

(3.28)

This list exhausts all highest weight $\mathfrak{sl}(11-D,\mathbb{R})$ representations present for $D = 3$. In the infinite-dimensional case, there is an infinite number of other $\mathfrak{sl}(11-D,\mathbb{R})$ representations. The question of their identification is still a largely open question. Progresses have been made lately in identifying some roots of E_{10} as one-loop corrections to $11D$ supergravity [18] or as Minkowskian M-branes and additional solitonic objects of M-theory [21]. These questions will be introduced in the next section, and will become one of the main topics of the last part of this paper.

However, it is worth noting that for $D \leqslant 2$, the 8-form generator is now subject to the Jacobi identity

$$\widetilde{K}^{[i_1|i_2..i_9]} = 0\,. \tag{3.29}$$

In $D = 2$, this reflects the fact that the would-be totally antisymmetric generator $\widetilde{K}^{[i_1 i_2..i_9]}$ attached to the null root δ is not the dual of a supergravity scalar, but corresponds to the root space $\{z \otimes H_i\}_{i=1,..,8}$, and reflects the localization of the U-duality symmetry.

In addition, we denote the compact generators by $\mathcal{K}_i^{\ j} = K_i^{+\,j} - K_i^{-\,j}$, $\mathcal{Z}^{ijk} = Z^{+\,ijk} - Z^{-\,ijk}$ and similarly for $\mathcal{Z}^{i_1..i_6}$ and $\widetilde{\mathcal{K}}_j^{i_1..i_8}$. Then:

$$K(G^U) = \mathrm{Span}\left\{\mathcal{K}_i^{\ j}; \mathcal{Z}^{ijk}; \mathcal{Z}^{i_1..i_6}; \widetilde{\mathcal{K}}_j^{\ i_1..i_8}\right\}$$

Fixing the normalization of the compact generators to 1 has been motivated by the algebraic orbifolding procedure we will use in the next sections, and ensures that automorphism generators and the orbifold charges they induce have the same normalization.

In particular, the compact Lorentz generators $\mathcal{K}_i^{\ i+1} \equiv E_{\alpha_{i-2}} - F_{\alpha_{i-2}}$, $\forall i = D, \ldots, 9$ clearly generate rotations in the $(i, i+1)$-planes, so that a general rotation in the (i,j)-plane is induced by $\mathcal{K}_i^{\ j}$. One can check that, as expected, $[\mathcal{K}_k^{\ j}, X_{i_1..j..i_p}] = X_{i_1..k..i_p}$ $\forall X \in$

$\mathfrak{g}^U/\mathfrak{k}(\mathfrak{g}^U)$ in Table (3.28). For instance, the commutator $[\mathcal{K}_i^{\ i+1}, Z_{i+1\,jk}]$ for $i+1<j<k$ belongs to the root space of:

$$\alpha = \alpha_{i-2} + .. + \alpha_{j-3} + 2(\alpha_{j-2} + .. + \alpha_{k-3}) + 3(\alpha_{k-2} + .. + \alpha_5) + 2\alpha_6 + \alpha_7 + \alpha_8$$

that defines $Z_{ijk} = (1/2)(E_\alpha + F_\alpha)$.

As a final remark, note that the group element $\widetilde{g}$ with value in G^U can be used to reinstate local $K(G^U)$-invariance of the algebraic field strength $\widetilde{g}^{-1}d\widetilde{g} = \mathcal{P} + \mathcal{Q}$ under the transformation as $\widetilde{g}(x) \to U\widetilde{g}(x)h(x)^{-1}$ for $h(x) \in K(G^U)$ and a rigid U-duality element $U \in G^U$. In this case, $\mathcal{Q}$ transforms as a generalized connection:

$$\mathcal{Q} \to h(x)\mathcal{Q}h(x)^{-1} - h(x)\,dh(x)^{-1}\,,$$

and $\mathcal{P}+\mathcal{Q}$ as a generalized field-strength: $\mathcal{P}+\mathcal{Q} \to U^{-1}(\mathcal{P}+\mathcal{Q})U$. Performing a level expansion of $\mathcal{Q}$:

$$\mathcal{Q} = \frac{1}{2}dx^A\left(\omega_{Aj}^{\ \ i}\mathcal{K}_i^{\ j} + \omega_A^{\ ijk}\mathcal{Z}_{ijk}\right) + \dots$$

we recognize for $l=0$ the Lorentz connection, for $l=1$ the 3-form gauge connection, etc.

Actually our motivation for working in the symmetric gauge comes from the fact that, at the level of the algebra, the orbifold charge operator acting as Adh preserves this choice. Indeed $h \in K(G^U)$ is in this case a rigid transformation, so that one can drop the connection part $\mathcal{Q}$ in expression (3.26), and Adh normalizes $\mathcal{P}$.

Along this line, a non-linear realization where only local Lorentz invariance is implemented has been used extensively in [54, 60, 61] to uncover very-extended Kac-Moody hidden symmetries of various supergravity theories. This has led to the conjecture that $\mathfrak{e}_{11}$ is a symmetry of $11D$ supergravity, and possibly M-theory, as this very-extended algebra can be obtained as the closure of the finite Borel algebra of a non-linear realization similiar to the one we have seen above, with the $11D$ conformal algebra.

3.4 M-theory near a space-like singularity as a $E_{10|10}/K(E_{10|10})$ σ-model

In the preceding section, we have reviewed some basic material about $11D$ supergravity compactified on square tori, which we will need in this paper to derive the residual U-duality symmetry of the untwisted sector of the theory when certain compact directions are taken on a orbifold. The extension of this analysis to the orbifolded theory in $D=2,1$ dimensions, where KM hidden symmetries are expected to arise, will require a generalization of the low-energy effective supergravity approach. The proper framework to treat hidden symmetries in $D=1$ involves a σ-model based on the infinite coset $E_{10|10}/K(E_{10|10})$. In the vicinity of a space-like singularity, this type of model turns out to be a generalization of a Kasner cosmology, leading to a null geodesic motion in the moduli space of the theory, interrupted by successive reflections against potential walls. This dynamics is usual referred to as a cosmological billiard, where by billiard, we mean a convex polyhedron with finitely many vertices, some of them at infinity.

In [15, 24] the classical dynamics of M-theory near a spacelike singularity has been conjectured to possess a dual description in terms of this chaotic hyperbolic cosmological

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billiard. In particular, these authors have shown that, in a small tension limit $l_p \to 0$ corresponding to a formal BKL expansion, there is a mapping[6] between (possibly composite) operators[7] of the truncated equations of motion of $11D$ supergravity at a given spatial point, and one-parameter quantities (coordinates) in a formal σ-model over the coset space $E_{10|10}/K(E_{10|10})$. More recently, [18] has pushed the analysis even further, and shown how higher order M-theory corrections to the low-energy $11D$ supergravity action (similar to α' corrections in string theory) are realized in the σ-model, giving an interpretation for certain negative imaginary roots of E_{10}.

In particular, the regime in which this correspondence holds is reached when at least one of the diagonal metric moduli is small, in the sense that $\exists i$ s.t. $R_i \ll l_P$. In this case, the contributions to the Lagrangian of $11D$ supergravity (with possible higher order corrections) coming from derivatives of the metric and p-form fields can be approximated by an effective potential, with polynomial dependence on the diagonal metric moduli. In the BKL limit, these potential terms become increasingly steep, and can be replaced by sharp walls or cushions, which, on the $E_{10|10}/K(E_{10|10})$ side of the correspondence, define a Weyl chamber of E_{10}. The dynamics of the model then reduces to the time evolution of the diagonal metric moduli which, in the coset, map to a null geodesic in the Cartan subalgebra of E_{10} deflected by successive bounces against the billiard walls. In the leading order approximation, one can restrict his attention to the dominant walls, *i.e.* those given by the simple roots of E_{10}, so that the billiard motion is confined to the fundamental Weyl chamber of E_{10}. As mentioned before, [15, 24, 18] have shown how to extend this analysis to other Weyl chambers by considering higher level non-simple roots of E_{10}, and how the latter can be related, on the supergravity side, to composite operators containing multiple gradients of the supergravity fields and to M-theory corrections. These higher order terms appear as one considers smaller and smaller corrections in l_P as we approach the singularity $x^0 \to \infty$. These corrections are of two different kind: they correspond either to taking into account higher and higher spatial gradients of the supergravity fields in the truncated equations of motion of $11D$ supergravity at a given point of space, or to considering M-theory corrections to the classical two-derivative Lagrangian.

In the following, since we ultimately want to make contact with [21, 19], we will consider the more restrictive case in which the space is chosen compact, and is in particular taken to be the ten-dimensional torus T^{10}, with periodic coordinates $0 \leqslant x^i < 2\pi$, $\forall i = 1, \ldots, 10$. This, in principle, does not change anything to the non-compact setup of [62], since there the mapping relates algebraic quantities to supergravity fields at a *given point* in space, regardless in principle of the global properties of the manifold.

Before tackling the full-fledged hyperbolic E_{10} billiard and the effective Hamiltonian description of 11D supergravity dual to it, it is instructive to consider the toy model obtained by setting all fields to zero except the dilatons. This leads to a simple cosmological model characterized by a space-like singularity at constant time slices t. This suggests to introduce a lapse function $N(t)$. The proper time σ is then defined as $d\sigma = -N(t)dt$,

[6]Which has been worked out up to order $l = 6$ and $\mathrm{ht}(\alpha) = 29$.

[7]Constructed from the vielbein, electric and magnetic components of the four-form field-strength, and their multiple spatial gradients.

and degenerates ($\sigma \to 0^+$) at the singularity $N(t) = 0$. This particular limit is referred to as the BKL limit, from the work of Belinskii, Khalatnikov and Lifshitz [16, 17]. As one approaches the singularity, the spatial points become causally disconnected since the horizon scale is smaller than their spacelike distance.

In this simplified picture, the metric (3.15) reduces to a Kasner one, all non-zero fields can be taken to depend only on time (since the space points are fixed):

$$ds^2 = -(N(t)dt)^2 + \sum_{i,j=1}^{10} e^{2\langle H_R(t),\varepsilon_i\rangle}\delta_{ij}dx^i \otimes dx^j \,. \tag{3.30}$$

In addition to proper time σ, we introduce an "intermediate time" coordinate u defined as

$$du = -\frac{1}{\sqrt{g}}d\sigma = \frac{N(t)}{M_P^{10}V(t)}dt\,, \tag{3.31}$$

where $\sqrt{g} = \sqrt{\det g_{ij}} = e^{\langle H_R,\rho_1\rangle}$ and $V(t) = \prod_{i=1}^{10} R_i(t)$, where ρ_1 is the "threshold" vector (3.16). In this frame, one approaches the singularity as $u \to +\infty$.

Extremizing $\int eR$ with respect to the R_i and $N/\sqrt{g}$, we get the equations of motion for the compactification radii and the zero mass condition:

$$\frac{d}{du}\left(\frac{1}{R_i}\frac{dR_i}{du}\right) = 0\,, \qquad \sum_i\left(\frac{\dot{R}_i}{R_i}\right)^2 - \left(\sum_i\frac{\dot{R}_i}{R_i}\right)^2 = 0\,, \tag{3.32}$$

where the dot denotes $\frac{d}{dt}$. Setting $R_i(u_0) = R_i(s_0) = M_P^{-1}$, one obtains R_i in terms either of u or of σ

$$M_P R_i = e^{-v_i(u-u_0)} = \left(\frac{\sigma}{\sigma_0}\right)^{\frac{v_i}{\sum_j v_j}} \tag{3.33}$$

since $u = -\frac{1}{\sum_j v_j}\ln(\sigma + const) + const'$. Then, the evolution of the system reduces to a null geodesic in $\mathfrak{h}(E_{10})$. In the u-frame in particular, the vector $H_R(u) = \sum_i \ln(M_P R_i(u))\varepsilon^{\vee i}$ can be regarded as a particle moving along a straight line at constant velocity $-\vec{v}$. In the u-frame, it is convenient to define $\vec{p} = (\sum_j v_j)^{-1}\vec{v}$, whose components are called Kasner exponents. These satisfy in particular:

$$\sum_i p_i^2 - \left(\sum_i p_i\right)^2 = 0\,, \qquad \sum_i p_i = 1\,. \tag{3.34}$$

The first constraint originates from the zero mass condition (3.32) and implies $\vec{p} \in \mathfrak{h}^*(E_{10})$, while the second one comes from the very definition of the p_i's. These two conditions result in at least one of the p_i's being positive and at least another one being negative, which leads, as expected, to a Schwartzchild type singularity in the far past and far future.

In the general case, we reinstate off-diagonal metric elements in the line-element (3.30) by introducing the vielbein (3.20) in triangular gauge:

$$\delta_{ij}dx^i \otimes dx^j \to \delta_{ij}\tilde{\gamma}^i{}_p\tilde{\gamma}^j{}_q dx^p \otimes dx^q\,, \tag{3.35}$$

with $\tilde{\gamma}^i{}_p = (\delta^i{}_p + \mathcal{A}^i{}_p)$, and $\mathcal{A}^i{}_p$ defined for $i < p$. For reasons of clarity, we discriminate this time the flat indices (i, j, k, l) from the curved ones (p, q, r, s).

In this more general case, it can be shown [62], that asymptotically (when approaching the singularity), the log of the scale factors $\ln M_P R_i$ are still linear functions of u, while the off-diagonal terms $\mathcal{A}^i{}_j$ tend to constants: in billiard language, they *freeze* asymptotically.

To get the full supergravity picture, one will in addition turn on electric 3-form and magnetic 6-form fields and the duals to the Kaluza-Klein vectors, and possibly other higher order corrective terms. Provided we work in the Iwasawa decomposition (3.15), one can show that, similarly to the off-diagonal metric components, these additional fields and their multiple derivatives also freeze as one approaches the singularity. In particular, all $(p+1)$-form field strengths will tend to constants in this regime, and therefore behave like potential terms for the dynamical scale factors.

An effective Hamiltonian description of such a system has been proposed [62, 18]:

$$\mathcal{H}(H_R, \partial_u H_R, F) = B(\partial_u H_R, \partial_u H_R) + \frac{1}{2} \sum_A e^{2w_A(H_R)} c_A(F)\,, \tag{3.36}$$

For later convenience, we want to keep the dependence on conformal time apparent, so that we use $\partial_u H_R$ to represent the canonical momenta given by $\pi^i = 2(g_\varepsilon^\vee)^{ij}\,\partial_u \ln M_P R_i$. In units of proper time (3.52), the Hamiltonian is then given by the integral:

$$H = \int d^{10}x \,\frac{N}{\sqrt{g}}\,\mathcal{H}\,. \tag{3.37}$$

Let us now discuss the structure of $\mathcal{H}$ (3.36) in more details. First, the Killing form B is defined as in eqn. (3.9) and is alternatively given by the metric $g_\varepsilon^\vee$. It determines the kinetic energy of the scale factors. The second term in expression (3.36) is the effective potential generated by the frozen off-diagonal metric components, the p-form fields, and multiple derivatives of all of them, which are collectively denoted by F. The (possibly) infinite sum over A includes the basic contributions from classical $11D$ supergravity (3.21), plus higher order terms related to quantum corrections coming from M-theory. In the vicinity of a spacelike singularity, the dependence on the diagonal metric elements factorizes, so that these contributions split into an exponential of the scale factors, $e^{2w_A(H_R)}$, and a part that freezes in this BKL limit, generically denoted by $c_A(F)$.

These exponential factors $e^{2w_A(H_R)}$ behave as sharp wall potentials, now interrupting the former straight line null geodesics $H_R(u)$ and reflecting its trajectory, while conserving the energy of the corresponding virtual particle and the components of its momentum parallel to the wall. In contrast, the perpendicular components change sign. Despite these reflections, the dynamics remains integrable and leads to a chaotic billiard motion. The reflections off the walls happen to be Weyl-reflections in $\mathfrak{h}(E_{10})$, and therefore conserve the kinetic term in $\mathcal{H}$ (3.36). However, since the Weyl group of E_{10} is a subgroup of the U-duality group, it acts non-trivially on the individual potential terms of $\mathcal{H}$. As the walls represent themselves Weyl reflections, they will be exchanged under conjugation by the Weyl group. More details on the action of the U-duality group in the general case, and in relation with hyperbolic billiard dynamics can be found in Appendix B.

In the BKL limit then, the potential terms $e^{2w_A(H_R)}$ can be mimicked by theta-functions: $\Theta\left(w_A(H_R)\right)$ so that the dynamics is confined to a billiard table defined by the inequalities $w_A(H_R) \leqslant 0$. If one can isolate, among them, a finite set of inequalities $I = \{A_1, .., A_n\}$, $n < \infty$, which imply all the others, the walls they are related to are called *dominant*.

The contributions to the effective potential in $\mathcal{H}$ (3.36) arising from classical supergravity can be described concretely, and we can give to the corresponding walls an interpretation in terms of roots of E_{10}. As a first example, we give the reduction on T^{10} of the kinetic energy for the 3-form potential, and write it in terms of the momenta conjugate to the C_{ijk}:

$$\frac{1}{2}\sum_{i<j<k} e^{2w_{ijk}(H_R)}(\underline{\pi}^{ijk})^2 = \frac{1}{2}\sum_{i<j<k}(M_P^3 R_i R_j R_k)^2\left[\tilde{\gamma}^i{}_p\tilde{\gamma}^j{}_q\tilde{\gamma}^k{}_r\,\pi^{pqr}\right]^2 . \tag{3.38}$$

As pointed out above, the momenta $\underline{\pi}^{ijk}$ freeze in the BKL limit. Their version in curved space can be computed to be

$$\pi^{p_1p_2p_3} = \sum_{i<j<k} e^{-2w_{ijk}(H_R)}\gamma^{p_1}{}_i\gamma^{p_2}{}_j\gamma^{p_3}{}_k\gamma^{q_1}{}_i\gamma^{q_2}{}_j\gamma^{q_3}{}_k\partial_u C_{q_1q_2q_3}\,,$$

with $\partial_u C_{q_1q_2q_3} = \sqrt{g}\,G_{0q_1q_2q_3}$, since the flat time-index is defined by: $dx^0 = N(t)dt$. From expression (3.38), one identifies the walls related to the three-form effective potential, and referred to as "electric" in [62], with $l = 1$ positive roots of E_{10}, namely: $w_{ijk} = \varepsilon_i + \varepsilon_j + \varepsilon_k \in W_{\mathrm{M2}}(E_{10})$.

Note, in passing, that the exponential in eqn.(3.38) has the opposite sign compared to the reduced Lagrangian (3.21) for $D \geqslant 3$. This is a consequence of opting for the Hamiltonian formalism, where the Legendre transform inverts the sign of the phase factor $e^{2w_A(H_R)}$ for the momenta π^{pqr}. In this respect, the latter are defined with upper curved indices (flattened by $\tilde{\gamma}^i{}_p$), as in expression (3.38), while their conjugate fields carry lower curved ones (flattened by the inverse $\gamma^i{}_p \doteq (\tilde{\gamma}^{-1})^i{}_p$, see (3.39) below). For more details, see [62]. In any case, one can simultaneously flip all signs in the wall factors for both the Lagrange and Hamiltonian formalisms, by choosing a lower triangular parametrization for the vielbein (3.35), which corresponds to an Iwasawa decomposition with respect to the set of negative roots of the U-duality group.

Similarly, there will be a potential term resulting from the dual six-form $\widetilde{C}_6$ kinetic term (the second term in the second line of expression (3.21)). In contrast to eqn. (3.21), we rewrite the electric field energy for $\widetilde{C}_6$ as the magnetic field energy for C_3:

$$\begin{aligned}&\frac{1}{2}\sum_{i<j<k<l} e^{2w_{ijkl}(H_R)}(\underline{G}_{ijkl})^2 = \\ &\quad\frac{1}{2}\sum_{i_1<\ldots<i_6}\sum_{i_7<\ldots<i_{10}}(M_P^6 R_{i_1}\cdots R_{i_6})^2\left[\gamma^p{}_{i_7}\gamma^q{}_{i_8}\gamma^r{}_{i_9}\gamma^s{}_{i_{10}}G_{pqrs}\,\epsilon^{i_1\ldots i_{10}}\right]^2\end{aligned} \tag{3.39}$$

Again, the components $\underline{G}_{ijkl}$ freeze in the BKL limit, leaving a dependence on the "magnetic" walls given by $l = 2$ roots of E_{10}: $w_{ijkl} = \sum_{m\notin\{i,j,k,l\}}\varepsilon_m \in W_{\mathrm{M5}}(E_{10})$. Dualizing this expression with respect to the ten compact directions, we can generate Chern-Simons terms resulting from the topological couplings appearing in the definition of $\underline{G}_{2\,ij}$ in eqn. (3.18), namely: $2\gamma^{p_1}_{i_1}\gamma^{p_2}_{i_2}\gamma^{p_3}{}_{i_3}\gamma^{p_4}_{i_4}\gamma^{p_5}_{i_5}G_{p_3p_4p_5[p_1}\mathcal{A}^{i_5}{}_{p_2]}$. However, such contributions are characterized by the same walls as expression (3.39), and thus have no influence on the asymptotic billiard dynamics, but only modify the constraints.

The off-diagonal components of the metric $\mathcal{A}^i{}_j$ will also contribute a potential term in $\mathcal{H}$(3.36). Inspecting the second line of expression (3.21), we recognize it as the frozen kinetic part of the first term on this second line:

$$\frac{1}{2}\sum_{i<j} e^{2w_{ij}(H_R)}(\underline{\pi}^i{}_j)^2 = \frac{1}{2}\sum_{i<j}\left(\frac{R_i}{R_j}\right)^2\left[\tilde{\gamma}^i{}_p\pi^p{}_j\right]^2 \tag{3.40}$$

where the momentum with curved indices is defined as

$$\pi^p{}_j = \sum_k e^{-2w_{kj}(H_R)}\gamma^p{}_k\gamma^r{}_j\partial_u\mathcal{A}^k{}_r\,, \qquad \text{with } k<j\,.$$

The sharp walls appearing in this case are usually called *symmetry* (or *centrifugal*) walls and correspond to $l=0$ roots of E_{10}, namely: $w_{ij}=\varepsilon_i-\varepsilon_j \in W_{\mathrm{KKp}}(E_{10})$.

Finally, the curvature contribution to the potential in $\mathcal{H}$ (3.36) produces two terms:

$$\begin{aligned}
&\frac{1}{2}\sum_{j<k}\sum_{i\neq\{j,k\}} e^{2\widetilde{w}_{ijk}(H_R)}(\underline{\mathcal{F}}^i{}_{jk})^2 - \sum_i e^{2w_i(H_R)}(\underline{\mathcal{F}}_i)^2\\
&= 2\sum_{i_1<\ldots<i_7,i_8}\ \sum_{i_9<i_{10}}(M_P^9R_{i_1}\cdots R_{i_7}R_{i_8}^2)^2(\gamma^p{}_{i_9}\gamma^q{}_{i_{10}}\partial_{[p}\mathcal{A}^{i_8}{}_{q]}\epsilon^{i_1\ldots i_{10}})^2\\
&\quad-\sum_{i_1<\ldots<i_9,i_{10}}(M_P^9R_{i_1}\cdots R_{i_9})^2(\underline{\mathcal{F}}_{i_{10}}\epsilon^{i_1\ldots i_{10}})^2\,.
\end{aligned} \tag{3.41}$$

The first one is already present as the third term on the second line of expression (3.21), the $\underline{\mathcal{F}}^i{}_{jk}$ being related to the spatial gradients of the metric, or, alternatively, to the structure functions of the Maurer-Cartan equation for the vielbein (3.35):

$$\underline{\mathcal{F}}^i{}_{jk} = 2\gamma^p_j\gamma^q{}_k\partial_{[p}\mathcal{A}^i{}_{q]} \tag{3.42}$$

As for expression (3.39), one can generate Chern-Simons couplings $\gamma^p_j\gamma^q{}_k\gamma^r_l\mathcal{F}^i{}_{r[p}\mathcal{A}^l{}_{q]}$ by dualizing the above expression in the ten compact directions. This again will not generate a new wall, and, as for expression (3.42), corresponds to $l=3$ roots of E_{10} given by $\widetilde{w}_{ijk}=\sum_{l\notin\{i,j,k\}}\varepsilon_l+2\varepsilon_i \in W_{\mathrm{KK7M}}(E_{10})$.

The $\underline{\mathcal{F}}_i$ on the other hand are some involved expressions depending on the fields R_i, ∂R_i, $\underline{\mathcal{F}}^i{}_{jk}$ and $\partial\underline{\mathcal{F}}^i{}_{jk}$. In eqn.(3.41), they are related to lightlike walls $w_i=\sum_{k\neq i}\varepsilon_k$ given by all permutations of the null root $\delta=(0,(1)^9)$. These prime isotropic roots are precisely the ones at the origin of the identity (3.29). Since they can be rewritten as $w_i=(1/2)(\widetilde{w}_{jki}+\widetilde{w}_{kij})$, they are subdominant with respect to the $\widetilde{w}_{ijk}$, and will not affect the dynamics of H_R even for $\vec{p}$ close to the lightlike direction they define. So they are usually neglected in the standard BKL approach. In the next section, we will see that these walls have a natural interpretation as Minkowskian KK-particles [21], and contribute matter terms to the theory.

All the roots describing the billiard walls we have just listed are, except for w_i, real $l \leqslant 3$ roots of E_{10}, and the billiard dynamics constrains the motion of H_R to a polywedge bounded by the hyperplanes: $\langle H_R(t), w_A \rangle = 0$, with A spanning the indices of the walls mentioned above. The dominant walls are then the simple roots of E_{10}. In this respect, the orbits $W_{\rm M5}(E_{10})$ and $W_{\rm KK7M}(E_{10})$ contains only subdominant walls, which are *hidden* behind the dominant ones, and can, in a first and coarse approximation, be neglected. The condition $\alpha_i(H_R(t)) \leqslant 0$, with $\alpha_i \in \Pi(E_{10}) \subset W_{\rm KKp}(E_{10}) \cup W_{\rm M2}(E_{10})$, leads to the constraints:

$$R_1 \leqslant R_2\,, \quad R_2 \leqslant R_3\,, \ldots, \quad R_9 \leqslant R_{10}\,, \quad \text{and } R_8 R_9 R_{10} \leqslant l_P^3 \tag{3.43}$$

and the motion on the billiard is indeed confined to the fundamental Weyl chamber of E_{10}. The order in expression (3.43) depends on the choice of triangular gauge for the metric (3.35), and does not hold for an arbitrary vielbein. In the latter case, the formal E_{10} coset σ-model is more complicated than expression (3.44) below.

At this stage, we can rederive the mapping between geometrical objects of M-theory on T^{10} and the formal coset σ-model on $E_{10|10}/K(E_{10|10})$ proposed by [15], for the first $l = 0, 1, 2, 3$ real positive roots of E_{10}. This geodesic σ-model is governed by the evolution parameter t, which will be identified with the time parameter (3.31). To guarantee reparametrization invariance of the latter, we introduce the lapse function n, different from N. Then, in terms of the rescaled evolution parameter $d\tau = ndt$, the formal σ-model Hamiltonian reads [62]:

$$\mathcal{H}(H_R, \partial_\tau H_R, \nu, \partial_\tau \nu) = n\Big(B(\partial_\tau H_R | \partial_\tau H_R) + \frac{1}{2} \sum_{\alpha \in \Delta_+(E_{10})} \sum_{a=1}^{m_\alpha} e^{2\langle H_R, \alpha\rangle} [P_{\alpha,a}(\nu, \partial_\tau \nu)]^2\Big) \tag{3.44}$$

where $(\nu, \partial_\tau \nu)$ denotes the infinitely-many canonical variables of the system. We again use $\partial_\tau H_R$ to represent the momenta $\pi^i = 2(g_\varepsilon^\vee)^{ij}\,(R_i)^{-1}\partial_\tau R_i$ conjugate to $\ln M_P R_i$. The metric entering the kinetic term is chosen to be $g_\varepsilon^\vee$, which is dictated by comparison with the bosonic sector of toroidally reduced classical 11D supergravity.

Expression (3.44) is obtained by computing the formal Lagrangian density from the algebraic field strength valued in $\mathfrak{a}(E_{10|10}) \oplus \mathfrak{n}(E_{10|10})$ as:

$$g^{-1}\frac{d}{dt} g = -\frac{1}{\sqrt{2}} \sum_i \frac{\dot{R}_i}{R_i}\, \varepsilon^{\vee i} + \sum_{\alpha \in \Delta_+(E_{10})} \sum_{a=1}^{m_\alpha} Y_{\alpha,a}(\nu, \dot{\nu})\, e^{-\langle H_R, \alpha\rangle} E^a_\alpha\,, \tag{3.45}$$

As in eq.(3.25), one starts by calculating $\mathcal{L} = n^{-1}\mathrm{Tr}(\mathcal{P}T(\mathcal{P}))$ with $\mathcal{P}$ given in expression (3.27). One then switches to the Hamiltonian formalism, with momentum-like variables given by the Legendre transform $P_{\alpha,a}(\nu, \dot{\nu}) = \frac{1}{n} e^{-2\langle H_R, \alpha\rangle} Y_{\alpha,a}(\nu, \dot{\nu})$, eventually leading to expression (3.44).

In the BKL limit, the (non-canonical) momenta tend to constant values $P_{\alpha,a}(\nu, \dot{\nu}) \to C_{\alpha,a}$, and the potential terms in expression (3.44) exhibit the expected sharp wall behaviour. One can now try and identifiy the roots $\alpha \in \Delta_+(E_{10})$ of the formal Hamiltonian (3.44) with the wall factors w_A in the effective supergravity Hamiltonian (3.36). With a consistent truncation to $l = 3$, for instance, one recovers the supergravity sector (3.24) on T^{10}. This corresponds to the mapping we have established between real simple roots of E_{10} and the symmetry, electric, magnetic and curvature walls w_{ij} w_{ijk}, w_{ijkl} and $\widetilde{w}_{ijk}$, which are all in $\Delta_+(E_{10})$. The identification of the algebraic coordinates $C_{\alpha,a}$ with geometrical objects in the low energy limit of M-theory given by $c_A(F)$ (as defined in (3.36) and below) can then be carried out.

Proceeding further to $l=6$, one would get terms related to multiple spatial gradients of supergravity fields appearing in the truncated equations of motion of $11D$ supergravity [15, 18] at a given point. Finally, considering a more general version of the Hamiltonian (3.44) by extending the second sum in the coset element (3.45) to negative roots, *i.e.*: $\alpha \in \Delta_+(E_{10}) \cup \Delta_-(E_{10})$, and pushing the level truncation to the range $l = 10$ to 28, one eventually identifies terms corresponding to 8th order derivative corrections to classical supergravity [18] of the form $R_2^m(DG_4)^n$, where R_2 is the curvature two-form and D is the Lorentz covariant derivative. At eighth order in the derivative, *i.e.* for $(m,n) \in \{(4,0),(2,2),(1,3),(0,4)\}$, they are typically related to $\mathcal{O}(\alpha'^3)$ corrections in $10D$ type IIA string theory, at tree level. In this case however, it may happen that the corresponding subleading sharp walls w_A are negative, which means that they can only be obtained for a non-Borel parametrization of the coset. In addition, they may not even be roots of E_{10}. However, these walls usually decompose into $w_A = -(n+m-1)\rho_1 + \zeta$, for $\zeta \in \Delta_+(E_{10})$, where the first term on the RHS represents the leading R^{n+m} correction. If $n+m = 3\mathbb{N}+1$, the $R^m(DF)^n$ correction under consideration is compatible with E_{10}, and ζ is regarded as the relative positive root associated to it.

This means that the w_A are not necessarily always roots of $\mathfrak{e}_{10}$, and when this is not the case, a (possibly infinite) subset of them can still be mapped to roots of E_{10}, by following a certain regular rescaling scheme.

3.5 Instantons, fluxes and branes in M-theory on T^{10}: an algebraic approach

If we now consider the hyperbolic U-duality symmetry E_{10} to be a symmetry not only of $11D$ supergravity, but also of the moduli space space of M-theory on T^{10}, which is conjectured to be the extension of expression (3.3) to $D=1$:

$$\mathcal{M}_{10} = E_{10|10}(\mathbb{Z})\backslash E_{10|10}/K(E_{10|10}) \tag{3.46}$$

the real roots appearing in the definition of the cosmological billiard are now mapped to totally wrapped Euclidean objects of M-theory, and can be identified by computing the action:

$$S_\alpha[M_P R_i] = 2\pi e^{\langle H_R,\alpha\rangle}\,, \qquad \alpha \in \Delta_+(E_{10})\,. \tag{3.47}$$

Thus, the roots of E_{10} found in the preceding section, namely: $w_{ij} = \varepsilon_i - \varepsilon_j \in W_{\mathrm{KKp}}(E_{10})$, $w_{ijk} = \varepsilon_i+\varepsilon_j+\varepsilon_k \in W_{\mathrm{M2}}(E_{10})$, $w_{i_1..i_6} = (\epsilon_{i_1}+..+\epsilon_{i_6}) \in W_{\mathrm{M5}}(E_{10})$ and $\widetilde{w}_{ijk} = \sum_{l\notin\{i,j,k\}}\varepsilon_l + 2\varepsilon_i \in W_{\mathrm{KK7M}}(E_{10})$ describe totally wrapped Euclidean Kaluza-Klein particles, M2-branes, M5-branes and Kaluza-Klein monopoles. The dictionary relating these roots of E_{10} to the action of extended objects of M-theory can be found in Table 1, for the highest weight of the corresponding representation of $\mathfrak{sl}(10,\mathbb{R})$ in $\mathfrak{e}_{10}$.

Now, as pointed out in [20], the (approximated) Kasner solution defines a past and future spacelike singularity. On the other hand, the low-energy limit in which $11D$ supergravity becomes valid requires all eleven compactification radii to be larger than l_P, and consequently the Kasner exponents to satisfy (for a certain choice of basis for $\mathfrak{h}(E_{10})$, which can always be made):

$$0 < p_{10} \leqslant p_9 \leqslant \ldots \leqslant p_1 \tag{3.48}$$

so that the vector $\overrightarrow{p}$ is timelike with respect to the metric $|\overrightarrow{p}|^2 = \sum_i p_i^2 - (\sum_i p_i)^2$ (3.9). Clearly, this does not satisfy the constraints (3.34) which require $\overrightarrow{p}$ to be lightlike. Such a modification of the Kasner solution (3.33) has been argued in [20] to be achieved if one includes matter, which dominates the evolution of the system in the infinite volume limit and thereby changes the solution for the geometry. This does not invalidate the Kasner regime prevailing close to the initial spacelike singularity, since, as pointed out in [20], matter and radiation become negligible when the volume of space tends to zero (even though their density becomes infinite). In the following, we will see how matter, in the form of Minkowskian particles and branes, have a natural interpretation in terms of some class of imaginary roots of $\mathfrak{e}_{10}$, and can thus be incorporated in the hyperbolic billiard approach.

In particular, the inequality (3.48) is satisfied if at late time we have

$$R_1 \gg R_2\,, \quad R_2 \gg R_3\,, \ldots,\ R_9 \gg R_{10}\,, \quad \text{and } R_8 R_9 R_{10} \gg l_P^3 \tag{3.49}$$

which can be rephrased as: $\langle H_R, \alpha_i \rangle \gg 0$, $\forall \alpha_i \in \Pi(\mathfrak{e}_{10})$. The action (3.47) related to such roots is then large at late time, and the corresponding Euclidean objects of Table 1 can then be used to induce fluxes in the background, and thus be related to an instanton effect. This is in phase with the analysis in [20], which states that at large volume, the moduli of the theory become slow variables (in the sense of a Born-Oppenheimer approximation) and can be treated semi-classically.

Let us now make a few remarks on the two different regimes encountered so far, the billiard and semi-classical dynamics. In the semi-classical regime of 3.49, we are well outside the fundamental Weyl chamber (3.43) and higher level roots of $\mathfrak{e}_{10}$ have to be taken into account and given a physical interpretation. In this limit of large radii, the dominance of matter and radiation will eventually render the dynamics non-chaotic at late times, but the vacuum of the theory can be extremely complicated, because of the presence of instanton effects and solitonic backgrounds. In contrast, in the vicinity of the spacelike singularity, matter and radiation play a negligible rôle, leading to the chaotic dynamics of billiard cosmology. On the other hand, the structure of the vacuum is simple in the BKL regime, in which the potential walls appear to be extremely sharp. It is characterized by ten flat directions bounded by infinite potential walls, the dominant walls of the fundamental Weyl chamber of $\mathfrak{e}_{10}$.

Finally, when $\overrightarrow{p}$ is timelike, it has been shown in [20] that the domain (3.48) where $11D$ supergravity is valid can be mapped, after dimensional reduction, to weakly coupled type IIA or IIB supergravity. For instance, the safe domain for type IIA string theory (where all the nine radii are large compared to l_s and $g_{IIA} < 1$, two parameters given in terms of $11D$ quantities in eqns.(3.54)) is given by:

$$p_{10} < 0 < p_{10} + 2p_9\,, \quad \text{and} \quad p_9 \leqslant p_8 \leqslant \ldots \leqslant p_1\,. \tag{3.50}$$

The two domains (3.48) and (3.50) are then related by U-duality transformations (cf. Appendix B).

Let us now discuss the issue of fluxes in this setup. From now on and without any further specification, we assume that the conditions (3.49) are met. Then, in addition to the instanton effects we have just mentioned, one can consider more complicated configurations by turning on some components of the p-form potentials of the theory. In this case, the action (3.47) receives an additional contribution due to the Wess-Zumino coupling of the p-form potential to the world-volume of the corresponding brane-like object. The action (3.47) will now receive a flux contribution which can be rephrased in algebraic terms as [63, 21]:

$$S_{\alpha_{(p)}}[M_P R_i; \mathcal{C}_{\alpha_{(p)}}] = 2\pi e^{\langle H_R, \alpha_{(p)}\rangle} + i\mathcal{C}_{\alpha_{(p)}} = \frac{M_P^{p+1}}{(2\pi)^p}\int_{\mathcal{W}_{p+1}} e\, d^{p+1}x + i\int_{\mathcal{W}_{p+1}} C_{p+1}\,, \tag{3.51}$$

where the $\alpha_{(p)}$ are positive real roots of $\mathfrak{e}_{10}$ given by the second column of Table 1, for all possible permutations of components in the physical basis. In particular, we will have three-form and six-form fluxes for non-zero potentials C_3 and $\widetilde{C}_6$ coupling to the Euclidean world-volumes $\mathcal{W}_3/\mathcal{W}_6$ of M2-/M5-branes respectively. For fluxes associated to Kaluza-Klein particle, we have the couplings $\mathcal{C}_{\alpha_{i-2}} = \int_\gamma g_{i\,i+1}/g_{i+1\,i+1}\, dx^i$, $i = 1,..,9$, where γ is the KK-particle world-line and the internal metric g can be written in terms of our variables R_i and $\mathcal{A}^i{}_j$ using equations (3.15) and (3.20). There is also a similar coupling of the dual potential $\widetilde{\mathcal{A}}^i{}_8$ to the eight-dimensional KK7M world-volume.

The moduli $M_P R_i$, $i = 1,..,10$, together with the fluxes from p-form potentials (3.51) parametrize the moduli space (3.46). Furthermore, on can define the following function which is harmonic under the action of a certain Laplace operator defined on the variables $\{M_P R_i; \mathcal{C}_\alpha\}^{i=1,..,10}_{\alpha\in\Delta_+(\mathfrak{e}_{10})}$ in the Borel gauge of $\mathfrak{e}_{10|10}$, and which is left-invariant under $E_{10|10}$:

$$\sqrt{N_p}\exp\left[-2\pi N_p\left(e^{\langle H_R,\alpha_{(p)}\rangle} \pm \frac{i}{2\pi}\mathcal{C}_{\alpha_p}\right)\right]\,.$$

In the limit of large radii, N_p is the instanton number and this expression is an extension to $\mathfrak{e}_{10}$ of the usual instanton corrections to string thresholds appearing in the low-energy effective theory. As such, it is conjectured to capture some of the non-perturbative aspects of M-theory [63].

Another kind of fluxes arise from non-zero expectation values of $(p+1)$-form field strengths. If we reconsider the effective Hamiltonian (3.36) in the region (3.49) where instanton effects are present, we notice that the action (3.47) appears in the effective potential as $\frac{1}{2\pi}S_\alpha = e^{2\alpha(H_R)}$. On the other hand, since their coefficients $c_A(F)$ freeze in the BKL limit, we may regard them as fluxes or topology changes provided the $(p+1)$-form field strengths appearing in eqns.(3.38), (3.39), (3.40) and (3.42) have, in this limit, integral background value:

$$\begin{aligned}
\underline{\pi}_{ijk} &\to (2\pi)^6\langle(*_{10}\underline{\pi})_{i_1..i_7}\rangle = \frac{1}{2\pi}\int_{\mathcal{C}_{i_1..i_7}} *_{10}\underline{\pi}_3 \in \mathbb{Z}\,,\\
\underline{G}_{ijkl} &\to (2\pi)^3\langle\underline{G}_{ijkl}\rangle = \frac{1}{2\pi}\int_{\mathcal{C}_{ijkl}} \underline{G}_4 \in \mathbb{Z}\,,\\
\underline{\pi}^i{}_j &\to (2\pi)^8\langle(*_{10}\underline{\pi}^i)_{j_1..j_9}\rangle = \frac{1}{2\pi}\int_{\mathcal{C}_{j_1..j_9}} *_{10}\underline{\pi}^i{}_1 \in \mathbb{Z}\,,\\
\underline{\mathcal{F}}^i{}_{jk} &\to 2\pi\langle\underline{\mathcal{F}}^i{}_{jk}\rangle = \int_{\mathcal{C}_{jk}} \mathrm{ch}_1\left(\underline{\mathcal{F}}^i{}_2\right) \equiv \mathrm{Ch}_1(\underline{\mathcal{F}}^i{}_2;\mathcal{C}_{jk}) \in \mathbb{Z}\,,
\end{aligned}$$

Where $\mathcal{C}_{i_1..i_{p+1}}$ is a $(p+1)$-cycle chosen along the appropriate spatial directions.

In particular, the coefficients $c_A(F)$ appearing in the potential terms (3.38) and (3.39) are now restricted to be integers: $c_A(F) \to \left[(2\pi)^6\langle(*_{10}\underline{\pi})_{i_1..i_7}\rangle\right]^2$ and $\left[(2\pi)^3\langle\underline{G}_{ijkl}\rangle\right]^2$ and generate respectively seven-form and four-form fluxes. In this perspective, the instantons encoded in the exponential term $e^{2w_A(H_R)} \equiv e^{2\langle H_R,\alpha_{(p)}\rangle}$ for $\alpha_{\mathrm{M5}} = \sum_{m\notin\{i,j,k,l\}}\varepsilon_m$ and $\alpha_{\mathrm{M2}} = \sum_{n\notin\{i_1,..,i_7\}}\varepsilon_n$ are understood as the process that changes the fluxes by an integral amount.

The wall coefficient $c_A(F) = \left[2\pi\langle\underline{\mathcal{F}}^i{}_{jk}\rangle\right]^2$ (3.42), on the other hand, corresponds to a deformation of the basic torus T^{10} to an S^1 fibration of the ith direction over the two-torus $T^2 = \{x^j, x^k\}$, in other words to the metric:

$$ds^2 = -(Ndt)^2 + \sum_{m\neq i}(M_P R_m)^2(dx^m)^2 + (M_P R_i)^2\left[dx^i - \frac{1}{2\pi}\mathrm{Ch}_1(\underline{\mathcal{F}}^i{}_2;\mathcal{C}_{jk})x^k dx^j\right]^2.$$

where the periodicity of x^k implies $x^i \to x^i + \mathrm{Ch}_1(\underline{\mathcal{F}}^i{}_2;\mathcal{C}_{jk})x^j$ for the fiber coordinate, all other coordinates retaining their usual 2π-periodicity. The value of $c_A(F)$ determines the first Chern character (or Chern class, since $\mathrm{ch}_1 = \mathrm{c}_1$) of the fibration, and the instanton effect associated to the root $\alpha_{\mathrm{KK7M}} = \sum_{l\notin\{i,j,k\}}\varepsilon_l + 2\varepsilon_i$ creates an integral jump in this first Chern number.

3.5.1 Minkowskian branes from prime isotropic roots of $\mathfrak{e}_{10}$

As mentioned in the preceding section, when considering the large volume limit (3.49) in the domain (3.48) where $11D$ supergravity is valid, one should in principle start considering higher level roots of $\mathfrak{e}_{10}$ in other Weyl chambers than the fundamental one. These roots, which, in the strict BKL limit, appear as subdominant walls and can be neglected in a first approximation, should now be taken into account as corrective or mass terms. In [21, 19], a program has been proposed to determine the physical interpretation of a class of null roots of $\mathfrak{e}_{10}$. These authors have, in particular, shown the correspondence between prime isotropic roots and Minkowskian extended objects of M-theory, for the first levels $l = 3, 5, 6, 7, 8$. On the other hand, as seen in Section 3.4, the autors of [18] have developed a different program where they identify imaginary (both isotropic and non-isotropic) bur also real roots, with $R^m(DF)^n$ type M-theory corrections to classical supergravity. However, these results have been obtained in an intermediate domain of the dynamical evolution, where not only negative roots become leading (accounting for the fact that these higher order corrections are described by negative roots), but where we expect quantum corrections to be visible. In the approach of [21], in contrast, the regime (3.49) allows for effects related to extended objects to become important, pointing, in the line of Section 3.5, at an interpretation for certain higher level roots in terms of branes and particles.

In the following, we give a condensed version of the correspondence between prime isotropic roots of $\mathfrak{e}_{10}$ and Minkowskian extended objects of M-theory, which can be found in a much more detailed and ample version in [21], which we follow closely until the end of this section.

First of all, since we now restrict to the region (3.48), we are sufficiently far from the singularity for the lapse function $N(t)$ to have any non-zero value. In particular, we can gauge-fix to $N(t) = M_P$ in expression (3.31), which defines the conformal time:

$$d\tilde{u} = \frac{dt}{M_P^9 V(t)}\,. \tag{3.52}$$

As we will see below, these are the "natural" units to work out the relation between prime isotropic roots of $\mathfrak{e}_{10}$ and Minkowskian particles and branes in M-theory.

Consider, for instance two M5 instantons at times $t_{\beta_{\rm M5}} \ll t_{\alpha_{\rm M5}}$ encoded algebraicly in

$$\alpha_{\rm M5} = ((1)^4, (0)^4, (1)^2), \qquad \beta_{\rm M5} = ((0)^4, (1)^6)$$

Since each of them creates a jump in their associated flux, inverting the time order to $t_{\beta_{\rm M5}} \gg t_{\alpha_{\rm M5}}$ will pass one instanton through the other, thereby creating a Minkowskian M2-brane stretched between them in the interval $[t_{\alpha_{\rm M5}}, t_{\beta_{\rm M5}}]$, where their respective fluxes overlap. This M2-brane will be associated to the root $\gamma_{\rm M2} = \alpha_{\rm M5} + \beta_{\rm M5} = ((1)^8, (2)^2)$. Recalling that we gauge-fixed to conformal time (3.52), the action for such an object has to be expressed in unit of conformal time, then:

$$\frac{d}{d\tilde{u}}\widetilde{S}_\alpha = M_P^9 V \frac{d}{dt} S_\alpha = 2\pi e^{\langle H_R, \alpha\rangle} \;\longrightarrow\; M_\alpha = \frac{1}{2\pi}\frac{d}{dt}S_\alpha = M_P^{-9} V^{-1} e^{\langle H_R, \alpha\rangle} \tag{3.53}$$

This expression for the mass of the object could also be deduced from the rescaling (3.37). Thus, in particular: $M_{\gamma_{\rm M2}} = M_P^3 R_9 R_{10}$ as expected from a membrane wrapped around directions x^9 and x^{10}.

From the supergravity perspective, the instanton described by $\alpha_{\rm M5}$ will create a jump in the flux: $(2\pi)^3\langle \underline{G}_{5678}\rangle \to (2\pi)^3 \langle \underline{G}_{5678}\rangle + 1$ when going from $t < t_{\alpha_{\rm M5}}$ to $t_{\alpha_{\rm M5}} < t$, while instanton $\beta_{\rm M5}$ induces $(2\pi)^3\langle \underline{G}_{1234}\rangle \to (2\pi)^3\langle \underline{G}_{1234}\rangle - 1$ when t passes $t_{\beta_{\rm M5}}$. Now the M2-brane flux sourced by $(2\pi)^6 \langle \underline{G}_{1234}\rangle\langle \underline{G}_{5678}\rangle$ via the topological term $\int C_3 \wedge G_4 \wedge G_4$ of $11D$ supergravity, has to be counterbalanced by an equal number of anti M2-branes. Thus, going from configuration $t_{\beta_{\rm M5}} \ll t_{\alpha_{\rm M5}}$ to configuration $t_{\beta_{\rm M5}} \gg t_{\alpha_{\rm M5}}$ after setting both initial fluxes to zero produces one unit of Minkowskian anti-M2-brane flux which must be compensated by the creation of a M2-brane in the same directions, as expected.

Another process involves passing an M2-instanton $\alpha_{\rm M2} = ((1)^2, (0)^7, 1)$ through a KK-monopole $\beta_{\rm KK7M} = ((0)^2, (1)^6, 2, 1)$. Since the KK-monopole shifts by one unit the first Chern class of the circular x^9-fibration over the (x^1, x^2)-torus: ${\rm Ch}_1(\underline{\mathcal{F}}^9{}_{(2)}; \mathcal{C}_{12}) \to{\rm Ch}_1(\underline{\mathcal{F}}^9{}_{(2)};$

$\mathcal{C}_{12}) + 1$, it creates an obstruction that blocks the M2-instanton somewhere in the (x^1, x^2) plane, and, by means of the fibration, produces an object at least wrapped along the x^9 direction. It is not hard to see that the Minkowskian object resulting from this process is a M2-brane wrapped around x^9 and the original x^{10}. One can check that the root $\alpha_{\rm M2} + \beta_{\rm KK7M} = \gamma_{\rm M2} = ((1)^8, (2)^2)$, recovering the same object as before.

Furthermore, by combining an M5-instanton $\alpha_{\rm M5} = (0, (1)^6, (0)^3)$ shifting the magnetic flux $(2\pi)^3\langle \underline{G}_{189\,10}\rangle$ by one unit with an M2-instanton $\beta_{\rm M2} = ((0)^7, (1)^3)$ shifting the electric flux $(2\pi)^6\langle (*_{10}\underline{\pi})_{1\cdots 7}\rangle$ accordingly, one creates a Minkowskian KK-particle $\alpha_{\rm KKp} = (0, (1)^9)$ corresponding to the following contribution the momentum in the x^1 direction: $\mathcal{P}^1 = \int d^{10}x\, G^{189\,10}\pi_{89\,10}$. The mass (3.53) of this object $M_{\gamma_{\rm KKp}} = R_1^{-1}$ is in accordance with this interpretation.

object	real root	S_α	prime isotropic	M_α
KKp	$(0,0,0,0,0,0,0,0,1,-1)$	$2\pi R_9 R_{10}^{-1}$	$(0,1,1,1,1,1,1,1,1,1)$	R_1^{-1}
M2	$(0,0,0,0,0,0,0,1,1,1)$	$2\pi M_P^3 R_8 R_9 R_{10}$	$(1,1,1,1,1,1,1,1,2,2)$	$M_P^3 R_9 R_{10}$
M5	$(0,0,0,0,1,1,1,1,1,1)$	$2\pi M_P^6 R_5\cdots R_{10}$	$(1,1,1,1,1,2,2,2,2,2)$	$M_P^6 R_5\cdots R_{10}$
KK7M	$(0,0,1,1,1,1,1,1,1,2)$	$2\pi M_P^9 R_3\cdots R_9 R_{10}^2$	$(1,1,1,2,2,2,2,2,2,3)$	$M_P^9 R_3\cdots R_9 R_{10}^2$
KK9M	$(1,1,1,1,1,1,1,1,1,3)$	$2\pi M_P^{12} R_1\cdots R_9 R_{10}^3$	$(1,2,2,2,2,2,2,2,2,4)$	$M_P^{12} R_2\cdots R_9 R_{10}^3$

Table 1: Euclidean and Minkowskian branes of M-theory on T^{10} and positive roots of E_{10}

Similar combinations of Euclidean objects can be shown, by various brane creation processes, to produce Minkowskian M5-branes and KK7M-branes. To conclude, all four types of time-extended matter fields are summarized in Table 1 by their highest weight representative with its mass formula. At present, it is still unclear how matter terms produced by the prime isotropic roots of Table 1 should be introduced in the effective Hamiltonian (3.36). Since the corresponding Minkowskian branes originate from creation processes involving two instantons, as explained above, we expect such a contribution to be [21]: $2\pi n_{\gamma_m} e^{\langle H_R, \gamma_m\rangle}$, where the isotropic root γ_m describing the Minkowskian brane decomposes into two real roots related to instantons $\gamma_m = \alpha_e + \beta_e$, and turns on $n_{\gamma_m} = n_{\alpha_e} n_{\beta_e}$ units of flux which compensates for the original n_{α_e} and n_{β_e} units of flux produced by the two instantons. Since γ_m is a root, such a term will never arise as a term in the serie (3.44). We then expect the Hamiltonian (3.44) to be modified in the presence of matter. In this respect, a proposal for a corrective term has been made in [21], which reproduces the energy of the Minkowskian brane only up to a 2π-factor. Moreover, it generates additional unwanted contributions for which one should find a cancelling mechanism.

From Table 1, one readily obtains the spectrum of BPS objects of type IIA string theory, by compactifying along x^{10}, and taking the limit $M_P R_{10} \to 0$, thereby identifying:

$$R_{10} = \frac{g_A}{M_s}\,, \qquad\qquad M_P = \frac{M_s}{\sqrt[3]{g_A}}\,. \tag{3.54}$$

In this respect, we have included in Table 1 the conjectured KK9M-brane as the putative M-theory ascendant of the D8, KK8A and KK9A-branes of IIA string theory, the latter two being highly non-perturbative objects which map, under T-duality, to the IIB S7 and S9-branes.

Since we mention type IIB string theory, we obtain its spectrum after compactification (3.54) by T-dualizing along the toroidal x^9 direction, which maps:

$$R_{9,B} = \frac{1}{M_s^2 R_{9,A}}\,, \qquad\qquad g_B = \frac{g_A}{M_s R_{9,A}}\,.$$

From the E_{10} viewpoint, T-dualizing from type IIA to IIB string theory corresponds to different embeddings of $\mathfrak{sl}(9,\mathbb{R})$ in $\mathfrak{e}_{10|10}$. Going back to the Dynkin diagram 1 for $r = 10$, type IIA theory corresponds to the standard embedding of $\mathfrak{sl}(9,\mathbb{R})$ along the preferred subalgebra $\mathfrak{sl}(10,\mathbb{R}) \subset \mathfrak{e}_{10|10}$ (the gravity line), while type IIB is obtained by choosing the the Dynkin diagram of $\mathfrak{sl}(9,\mathbb{R})$ to extend in the α_8 direction. This two embeddings consist in the two following choices of basis of simple roots for the Dynkin diagram of $\mathfrak{sl}(9,\mathbb{R})$:

$$\Pi_A = \{\alpha_{-1}, \alpha_0, \ldots, \alpha_5, \alpha_6\}\,, \qquad\qquad \Pi_B = \{\alpha_{-1}, \alpha_0, \ldots, \alpha_5, \alpha_8\}\,,$$

which results in two different identifications of the NS-NS sector of both theories. Then, a general T-duality on x^i, $i \neq 10$, can be expressed in purely algebraic language, as the following mapping:

$$\begin{gathered}\mathcal{T}_i : \mathfrak{h}(E_r) \to \mathfrak{h}(E_r)\\ \varepsilon_i \mapsto -\varepsilon_i\end{gathered}$$

From the σ-model point of view, the type IIA and IIB theories correspond to two different level truncations of the algebraic field strength (3.45), namely a level decomposition with respect to (l_7, l_8) for type IIA, and (l_6, l_7) for type IIB, l_i being the level in the simple root α_i of E_{10}. The NS-NS and RR sectors of both supergravity theories are then obtained by pushing the decomposition up to level $(l_7, l_8) = (2, 3)$ for IIA, and up to $(l_6, l_7) = (4, 2)$ for IIB. See for instance [64], where the results are directly transposable to E_{10} (all roots considered there are in fact E_{10} roots).

3.5.2 Minkowskian objects from threshold-one roots of $\mathfrak{e}_{10}$

By inspecting the second column of Table 1, we observe that all Minkowskian objects extended in p spatial directions, are characterized, on the algebraic side, by adding $\sum_{i=1}^{p} \varepsilon_{k_i}$ to the threshold vector $\rho_1 = ((1)^{10})$. For the Minkowskian KK particle, the corresponding root of $\mathfrak{e}_{10}$ is related to its quantized momentum, and one needs therefore to substract a factor of ε_j to the threshold vector.

The Minkowskian world-volume of these objects naturally couples to the respective $(p+1)$-form potentials (3.47). So, in contrast to the Hamiltonian formalism (3.36), which treats, for a different purpose, the temporal components of the field-strengths as conjugate momenta, one now needs to keep the time index of the tensor potentials apparent, thereby working in the Lagrange formalism. This is similar to what is done in [18] where the authors perform a component analysis of one-loop corrections to classical $11D$ supergravity.

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As pointed out in Section 3.4, the $(p+1)$-form components separate into an oscillating part and a part that freezes as $u \to +\infty$, so that we have:

$$C_{0i_1..i_q} = \frac{1}{N(t)} e^{p_1}{}_{i_1} \cdots e^{p_q}{}_{i_q} C_{tp_1\cdots p_q} = e^{-\langle \rho_1 + \sum_{n=1}^{q} \varepsilon_{i_n}, H_R\rangle} \gamma^{p_1}{}_{i_2} \cdots \gamma^{p_q}{}_{i_q} C_{up_1\cdots p_q}\,. \tag{3.55}$$

where we have used $\sqrt{g} = e^{\langle \rho_1, H_R\rangle}$, and the index 0 stands for the flat time coordinate $dx^0 = N(t)dt$. Following the analysis of Section 3.4, the component $C_{up_1\cdots p_q}$ can be shown to freeze. Now, by selecting the appropriate basis vectors ε_{j_n}, we observe that all imaginary roots in the second column of Table 1 are related to a tensor component of the form (3.55) with the expected value of q. As a side remark, the minus sign appearing in the exponential wall factor in eqn.(3.55) comes from working in the Lagrange formalism, as discussed in Section 3.5.

For Minkowskian KK particles and M2-branes, this approach is related to performing the $\mathfrak{e}_{10}$ extension of the last two sets of roots in eqns.(3.19) by setting $D = 1$. When restricting to the roots obtained by this procedure which are highest weights under the Weyl group of $\mathfrak{sl}(10, \mathbb{R})$, one again recovers the two first terms in the second column of Table 1. Since we have not performed any Hodge duality in this case, we obtain, as expected, roots characterizing KK particles and M2-branes.

It results from this simple analysis that it is the presence of the threshold vector $\rho_1 = ((1)^{10})$ which determines if an object is time-extended, and not necessarily the fact that the corresponding root is isotropic. We shall see in fact when working out 0B' orientifolds that certain types of magnetized Minkowskian D-branes can be associated to real roots and non-isotropic imaginary roots, provided they decompose as $\alpha = \rho_1 + \vec{q} \in \Delta_+(E_{10})$, where $\vec{q}$ is a positive vector (never a root) of threshold 0, *i.e.* that can never be written as $\vec{q} = n\rho_1 + \vec{q}'$ for $n \neq 0$.

4. Orbifolding in a KMA with non-Cartan preserving automorphisms

In this section, we expose the method we use to treat physical orbifolds algebraicly. It is based on the simple idea that orbifolding a torus by $\mathbb{Z}_n$ is geometrically equivalent to a formal $2\pi/n$ rotation. Using the mapping between physical and algebraic objects, one can then translate the geometrical rotation of tensors into purely algebraic language as a formal rotation in the KM algebra. This is given by an adjoint action of the group on its KM algebra given by a finite-order inner automorphism.

More concretely, let us consider an even orbifold T^q/Z_n, acting as a simultaneous rotation of angle $2\pi Q_a/n$, $a = 1, \ldots, q/2$ in each pair of affected dimensions determined by the charges $Q_a \in \{1, \ldots, n-1\}$. A rotation in the (x^i, x^j) plane is naturally generated by the adjoint action of the compact group element $\exp(\frac{2\pi}{n} Q_a \mathcal{K}^j{}_i) \equiv \exp(\frac{2\pi}{n} Q_a (E_\alpha - F_\alpha))$ for $\alpha = \alpha_{i-2} + \ldots + \alpha_{j-3}$. In particular, rotations on successive dimensions $(i+2, i+3)$ are generated by $E_{\alpha_i} - F_{\alpha_i}$. We will restrict ourselves to orbifolds acting only on successive pair of dimensions in the following, although everything can be easily extended to the general case. In particular, physically meaningful results should not depend on that choice, since it only amounts to a renumbering of space-time dimensions. For the same reason, we can restrict our attention to orbifolds that are taken on the last q spatial dimensions of space-time $\{x^{11-q}, \ldots, x^{10}\}$. In that case, we have the $q/2$ rotation operators

$$\begin{aligned} V_1 &\doteq e^{\frac{2\pi}{n} Q_1 \mathcal{K}_{11-q\,12-q}} = e^{\frac{2\pi}{n} Q_1 (E_{\alpha_{9-q}} - F_{\alpha_{9-q}})} \\ &\vdots \\ V_{\frac{q}{2}} &\doteq e^{\frac{2\pi}{n} Q_{\frac{q}{2}} \mathcal{K}_{9\,10}} = e^{\frac{2\pi}{n} Q_{q/2} (E_{\alpha_7} - F_{\alpha_7})} \end{aligned} \tag{4.1}$$

that all mutually commute, so that the complete orbifold action is given by:

$$\mathcal{U}_q^{\mathbb{Z}_n} \doteq \prod_{a=1}^{q/2} V_a\,. \tag{4.2}$$

Note that $\mathcal{U}_q^{\mathbb{Z}_n}$ generically acts non-trivially only on generators $\in \mathfrak{g}_{\pm\alpha}$ (and the corresponding $H_\alpha \in \mathfrak{h}$) for which the decomposition of α in simple roots contains at least one of the root $\alpha_{8-q}, \ldots, \alpha_8$ for $q > 2$ (α_6 and α_7 for $q = 2$).

In the particular case of $\mathbb{Z}_2$ orbifolds, the orbifold action leaves the Cartan subalgebra invariant, so that it can be expressed as a chief inner automorphism by some adjoint action $\exp(i\pi H)$, $H \in \mathfrak{h}_{\mathbb{Q}}$ on $\mathfrak{g}$. It indeed turns out that the action of such an automorphism on $\mathfrak{g}/\mathfrak{h} = \bigoplus_{\alpha\in\Delta(\mathfrak{g})} \mathfrak{g}_\alpha$ depends linearly on the root α grading $\mathfrak{g}_\alpha$ and can thus be simply expressed as:

$$\begin{aligned} \mathrm{Ad}(e^{i\pi H})\mathfrak{g}_\alpha = (-1)^{\alpha(H)}\mathfrak{g}_\alpha = (-1)^{\sum_{i=9-r}^{8} k^i \alpha_i(H)}\mathfrak{g}_\alpha\,, \\ \text{for } H \in \mathfrak{h}_{\mathbb{Q}} \ \text{ and } \ \alpha = \textstyle\sum_{i=9-r}^{8} k^i \alpha_i\,. \end{aligned} \tag{4.3}$$

If q is even, both methods are completely equivalent, while, if q is odd, the determinant is negative and it cannot be described by a pure $SO(r)$ rotation. If one combines the action of $E_\alpha - F_\alpha$ with some mirror symmetry, however, one can of course reproduce the action (4.3). Indeed, the last form of the orbifold action in expression (4.3) is the one given in [19]. The results of [19] are thus a subset of those obtainable by our more general method.

4.1 Cartan involution and conjugation of real forms

In the preceding sections, we have already been acquainted with the Chevalley involution ϑ_C. Here, we shall introduce just a few more tools we shall need later in this work to deal with real forms in the general sense. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. If it is related to a real Lie algebra $\mathfrak{g}_0$ as $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}} \doteq \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$, $\mathfrak{g}$ will be called a complexification thereof. Reciprocally, $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$ with $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$. Next, a semisimple real Lie algebra is called compact if it can be endowed with a Killing form satisfying

$$B(X, X) < 0\,, \quad \forall X \in \mathfrak{g}_0 \quad (X \neq 0)\,, \tag{4.4}$$

and non-compact otherwise.

Thus, a non-compact real form can in general be obtained from its complexification $\mathfrak{g}$ by specifying an involutive automorphism ϑ defined on $\mathfrak{g}_0$, such that

$$B_\vartheta(X, Y) \doteq -B(X, \vartheta Y) > 0\,, \quad \forall X, Y \in \mathfrak{g}_0 \quad (X, Y \neq 0)$$

called a Cartan involution (the argument here is a generalization of the construction of the almost-positive definite covariant form based on the Chevalley involution ϑ_C in Section 2.1). It can be shown that every real semisimple Lie algebra possesses such an involution, and that the latter is unique up to inner automorphisms. This is a corollary of the following theorem:

Theorem 4.1 *Every automorphism ψ of $\mathfrak{g}$ is conjugate to a chief automorphism ϑ of $\mathfrak{g}$ through an inner automorphism ϕ, ie:*

$$\psi = \phi^{-1} \circ \vartheta \circ \phi, \qquad \phi \in \mathrm{Int}(\mathfrak{g}) \tag{4.5}$$

Then, it is clear that ψ is involutive iff ϑ is involutive. In this case, the two real forms of $\mathfrak{g}$ generated by ψ and ϑ are isomorphic, so that for every conjugacy class of involutive automorphisms, one needs only consider the chief involutive automorphism (as class representative), which can in turn be identified with the Cartan involution.

The Cartan involution induces an orthogonal $(\pm 1)-$eigenspace decomposition into the direct sum $\mathfrak{g}_0 = \mathfrak{k} \oplus^{\perp} \mathfrak{p}$, called Cartan decomposition of $\mathfrak{g}_0$, with property

$$\vartheta|_{\mathfrak{k}} = 1 \text{ and } \vartheta|_{\mathfrak{p}} = -1 \ . \tag{4.6}$$

More specifically, $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}_0$ while $\mathfrak{p}$ is a representation of $\mathfrak{k}$, since: $[\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}] \subseteq \mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{k}$. Finally, as our notation for the Cartan decomposition suggested, $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal with respect to the Killing form and B_{ϑ}.

Alternatively, it is sometimes more convenient to define a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ as the fixed point subalgebra of $\mathfrak{g}$ under an involutive automorphism called conjugation τ such that

$$\tau(X) = X\,, \qquad \tau(iX) = -iX\,, \quad \forall X \in \mathfrak{g}^U \tag{4.7}$$

then: $\mathfrak{g}_0 = \{X \in \mathfrak{g} \,|\, \tau(X) = X\}$.

Finally, by Wick-rotating $\mathfrak{p}$ in the Cartan decomposition of $\mathfrak{g}_0$ one obtains the compact Lie algebra $\mathfrak{g}_c = \mathfrak{k} \oplus^{\perp} i\mathfrak{p}$ which is a compact real form of $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$.

Because of Theorem 4.1, one needs an invariant quantity sorting out involutive automorphisms leading to isomorphic real forms. This invariant is the signature (or character of the real Lie algebra) σ, defined as the difference between the number $d_- = \dim \mathfrak{k}$ of compact generators and the number $d_+ = \dim \mathfrak{p}$ of non-compact generators (the $\pm$-sign recalling the sign of the Killing form):

$$\sigma = d_+ - d_- \,.$$

For *simple* real Lie algebras, σ *uniquely* specifies $\mathfrak{g}_0$. The signature varies between its maximal value for the split form $\sigma = r$ and its minimal one for the compact form $\sigma = -\dim \mathfrak{g}$.

Defining the following linear operator constructed from ϑ (see [65], p.543)

$$\sqrt{\vartheta} = \frac{1}{2}(1+i)\vartheta + \frac{1}{2}(1-i)\mathbb{1}\ , \tag{4.8}$$

satisfying $\sqrt{\vartheta} \circ \sqrt{\vartheta}\, X = \vartheta X$, $\forall X \in \mathfrak{g}_c$, all non-compact real forms of $\mathfrak{g}$ will be obtained through

$$\mathfrak{g}_0 = \sqrt{\vartheta}\, \mathfrak{g}_c \tag{4.9}$$

by selecting the appropriate *chief* involutive automorphism ϑ.

4.2 Determining the real invariant subalgebra from its complexification

For a given orbifold $T^{11-q-D} \times T^q/\mathbb{Z}_n$ of eleven-dimensional supergravity/M-theory, the orbifold action on the corresponding U-duality algebra in D dimensions is given by the inner automorphism $\mathcal{U}_q^{\mathbb{Z}_n}$, $\forall D$. This automorphism has a natural extension to the complexification $(\mathfrak{g}^U)^{\mathbb{C}}$ of the split form $\mathfrak{g}^U$, where the appropriate set of generators describing physical fields and duality transformations on a complex space can be properly defined.

The requirement that these new generators diagonalize $\mathcal{U}_q^{\mathbb{Z}_n}$ and are charged according to the index structure of their corresponding physical objects will select a particular complex basis of $(\mathfrak{g}^U)^{\mathbb{C}}$. We will henceforth refer to this algebraic procedure as "orbifolding the theory".

Projecting out all charged states under $\mathcal{U}_q^{\mathbb{Z}_n}$ is then equivalent to an orbifold projection in the U-duality algebra, resulting in the invariant subalgebra $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}} = \text{Fix}_{\mathcal{U}_q^{\mathbb{Z}_n}}(\mathfrak{g}^U)^{\mathbb{C}}$ (the notation $\text{Fix}_V\mathfrak{g}$ stands for the fixed-point subalgebra of $\mathfrak{g}$ under the automorphism V).

Since we expect the untwisted sector of the theory to be expressible from the non-linear realization of $G_{\text{inv}}/K(G_{\text{inv}})$ as a coset sigma-model, we are particularly interested in determining the reality properties of $\mathfrak{g}_{\text{inv}}$, the algebra that describes the residual U-duality symmetry of the theory.

Retrieving the real form $\mathfrak{g}_{\text{inv}}$ from its complexification $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$ can be achieved by restricting the conjugation (4.7) to $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$. Denoting such a restriction $\tau_0 \doteq \tau|_{\mathfrak{g}_{\text{inv}}}$, the real form we are looking for is given by

$$\mathfrak{g}_{\text{inv}} = \text{Fix}_{\tau_0}({\mathfrak{g}_{\text{inv}}}^{\mathbb{C}})\,.$$

Since $\mathfrak{g}^U$ is naturally endowed with the Chevalley involution ϑ_C, the Cartan involution associated to the real form $\mathfrak{g}_{\text{inv}}$ is then the restriction of ϑ_C to the untwisted sector of the U-duality algebra, which we denote $\phi = \vartheta_C|_{\mathfrak{g}_{\text{inv}}}$. Consequently, the real form $\mathfrak{g}_{\text{inv}}$ possesses a Cartan decomposition $\mathfrak{g}_{\text{inv}} = \mathfrak{k}_{\text{inv}} \oplus \mathfrak{p}_{\text{inv}}$, with eigenspaces $\phi(\mathfrak{k}_{\text{inv}}) = \mathfrak{k}_{\text{inv}}$ and $\phi(\mathfrak{p}_{\text{inv}}) = -\mathfrak{p}_{\text{inv}}$. The whole procedure outlined in this section can be summarized by the following sequence:

$$(\mathfrak{g}^U, \vartheta_C) \xrightarrow{\mathcal{U}_q^{\mathbb{Z}_n}} (\mathfrak{g}^U)^{\mathbb{C}} \xrightarrow{\text{Fix}_{\mathcal{U}_q^{\mathbb{Z}_n}}} (\mathfrak{g}_{\text{inv}})^{\mathbb{C}} \xrightarrow{\text{Fix}_{\tau_0}} (\mathfrak{g}_{\text{inv}}, \phi)\,. \tag{4.10}$$

4.3 Non-compact real forms from Satake diagrams

As we have seen before, real forms are described by classes of involutive automorphisms, rather than by the automorphisms themselves. As such, the Cartan involution, which we will refer to as ϑ, can be regarded as some kind of preferred involutive automorphism, and is encoded in the so-called *Satake* diagram of the real form it determines. The Cartan involution splits the set of simple roots Π into a subset of black (invariant) roots ($\vartheta(\alpha_i) = \alpha_i$) we call Π_c, and the subset $\Pi_d = \Pi \,/\, \Pi_c$ of white roots, such as

$$\vartheta(\alpha_i) = -\alpha_{p(i)} + \sum_k \eta_{ik}\alpha_k, \text{ with } \alpha_{p(i)} \in \Pi_d \text{ and } \alpha_k \in \Pi_c$$

where p is an involutive permutation rotating white simple roots into themselves and η_{ik} is a matrix of non-negative integers. A Satake diagram consists in the Dynkin diagram of the complex form of the algebra with nodes painted in white or in black according to the above prescription. Moreover, if two white roots are exchanged under p, they will be joined on the Satake diagram by an arrow.

From the action of ϑ on the root system, one can furthermore determine the Dynkin diagram and multiplicities of the so-called restricted roots, which are defined as follows: for a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{p}$, let $\mathfrak{a} \subset \mathfrak{p}$ be maximal abelian. Then, one can define the partition under $\mathfrak{a}$ into simultaneous orthogonal eigenspaces (see [66] for a detailed discussion):

$$(\mathfrak{g}_0)_{\bar{\alpha}} = \{X \in \mathfrak{g}_0 \,|\, \mathrm{ad}(H_{\mathfrak{a}})X = \bar{\alpha}(H_{\mathfrak{a}})X, \ \forall H_{\mathfrak{a}} \in \mathfrak{a}\} \ . \tag{4.11}$$

This defines the restricted roots $\bar{\alpha} \in \mathfrak{a}^*$ as the simultaneous eigenvalues under the commuting family of self-adjoint transformations $\{\mathrm{ad}(H_{\mathfrak{a}})|\forall H_{\mathfrak{a}} \in \mathfrak{a}\}$. Then, we can choose a basis such that $\mathfrak{h}_0 = \mathfrak{t} \oplus \mathfrak{a}$, where $\mathfrak{t}$ is the maximal abelian subalgebra centralizing $\mathfrak{a}$ in $\mathfrak{k}$. The Cartan subalgebra can be viewed as a torus with topology $(S^1)^{n_c} \times (\mathbb{R})^{n_s}$ where $n_s = \dim \mathfrak{a}$ is called the $\mathbb{R}$-rank. Restricted-root spaces are the basic ingredient of the Iwasawa decomposition, so we shall return to them when discussing non-linear realizations (see Section 3.2) of orbifolded $11D$ supergravity/M-theory models.

We denote by Σ the set of roots not restricting to zero on $\mathfrak{a}^*$. As an example, one can choose a basis where such a set Σ reads:

$$\Sigma = \left\{ \bar{\alpha} = \frac{1}{2}(\alpha - \vartheta(\alpha)) \in \mathfrak{a}^* \,|\, \bar{\alpha} \neq 0 \right\} \ . \tag{4.12}$$

Then, a real form can be encoded in the triple $(\mathfrak{a}, \Sigma, m_{\bar{\alpha}})$ and $m_{\bar{\alpha}}$ is the function giving the multiplicity of each restricted root, in other words $m_{\bar{\alpha}} = \dim(\mathfrak{g}_0)_{\bar{\alpha}}$. If we denote by $\overline{\Pi}$ a basis of Σ, all non-compact real forms of $\mathfrak{g}$ can be encoded graphically in

I) the Satake diagram of (Π, ϑ);

II) the Dynkin diagram of $\overline{\Pi}$;

III) the multiplicities $m_{\bar{\alpha}_i}$ and $m_{2\bar{\alpha}_i}$ for $\bar{\alpha}_i \in \bar{B}$.

On the other hand, given a Satake diagram, we can determine the real form associated to it as a fixed point algebra under τ. Indeed from the Satake diagram one readily determines ϑ, and since it can be shown that there always exists a basis of $\mathfrak{h}$ such that the "compact" conjugation $\tau^c = \vartheta \circ \tau$ acts as $\tau^c(\alpha) = -\alpha$, $\forall \alpha \in \Delta$, then the conjugation is determined by $\tau = -\vartheta$ on the root lattice.

Finally, in the finite case, the $\mathbb{R}$-rank n_s is given in the Satake diagram by the number of white roots minus the number of arrows, and n_c by the number of black roots plus the number of arrows.

5. The orbifolds $T^2/\mathbb{Z}_{n>2}$

From the algebraic method presented in Section 4.2, it is evident that a $T^2/\mathbb{Z}_n$ orbifold on the pair of spatial dimensions $\{x^9, x^{10}\}$ is only expected to act non-trivially on the root spaces $(\mathfrak{g}^U)_\alpha \subset \mathfrak{g}^U$, characterized by all roots α containing α_6 and/or α_7, as well as on the corresponding Cartan element H_α.

The basis of $(\mathfrak{g}^U)^{\mathbb{C}}$ diagonalizing the orbifold automorphism $\mathcal{U}_2^{\mathbb{Z}_n}$ with the appropriate set of charges will be derived step by step for the chain of compactification ranging from $D = 8$ to $D = 1$. This requires applying the machinery of Section 4.2 to the generators of the root spaces $(\mathfrak{g}^U)_\alpha$ mentioned above and selecting combinations thereof to form a basis of $(\mathfrak{g}^U)^{\mathbb{C}}$ with orbifold charges compatible with their tensorial properties. We will at the same time determine the real invariant subalgebra $\mathfrak{g}_{\text{inv}}$ by insisting on always selecting lowest-height invariant simple roots, which ensures that the resulting invariant subalgebra is maximal. In $D = 2, 1$, subtleties connected with roots of multiplicities greater than one and the splitting of their corresponding root spaces will be adressed.

For a start, we will work out the $D = 8$ case in detail, and then show how this construction can be regularly extended down to the $D = 3$ case. The affine and hyperbolic $D = 2, 1$ cases require more care and will be treated separately. In $D = 8$, then, we consider eleven-dimensional supergravity on T^3, which possesses U-duality algebra $\mathfrak{g}^U = \mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$, whose complexification is described by the Dynkin diagram of $\mathfrak{a}_2 \oplus \mathfrak{a}_1$. It has positive-root space $\Delta_+ = \{\alpha_6, \alpha_7, \alpha_6 + \alpha_7, \alpha_8\}$, and its Cartan subalgebra is spanned by $\{H_6 = \varepsilon_8^\vee - \varepsilon_9^\vee\,;\, H_7 = \varepsilon_9^\vee - \varepsilon_{10}^\vee\,;\, H_8 = (2/3)(\varepsilon_8^\vee + \varepsilon_9^\vee + \varepsilon_{10}^\vee)\}$. The $\mathfrak{a}_1$ factor corresponds to transformations acting on the unique scalar $C_{89\,10}$ produced by dimensional reduction of the 3-form field on T^3.

The orbifold action on the two-torus

$$(z, \bar{z}) \to (e^{2\pi i/n} z, e^{-2\pi i/n} \bar{z}) \tag{5.1}$$

induces the following inner automorphism on the U-duality algebra

$$\mathcal{U}_2^{\mathbb{Z}_n} = \mathrm{Ad}(e^{\frac{2\pi}{n}(E_7 - F_7)}) \doteq \mathrm{Ad}(e^{-\frac{2\pi}{n} i\mathcal{K}_{z\bar{z}}})\,, \qquad \text{for } n > 2\,.$$

This automorphism acts diagonally on the choice of basis for $(\mathfrak{g}^U)^{\mathbb{C}}$ appearing in Table 2, where both compact and non-compact generators have the charge assignment expected from their physical counterparts.

The invariant subalgebra $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$ can be directly read off Table 2, the uncharged objects building an $\mathfrak{a}_1 \oplus \mathbb{C}^{\oplus^2}$ subalgebra, since the original $\mathfrak{a}_2$ factor of $(\mathfrak{g}^U)^{\mathbb{C}}$ now breaks into two abelian generators $H^{[2]} \doteq 2H_6 + H_7$ and $\widetilde{H}^{[2]} \doteq -\mathcal{K}_{z\bar{z}}$. The total rank (here 3) is conserved, which will appear to be a generic feature of $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$.

The real form $\mathfrak{g}_{\text{inv}}$ is then easily identified by applying the procedure outlined in eq. (4.10). Since $K_{z\bar{z}}$ and K_{88} are already in $\mathrm{Fix}_{\tau_0}(\mathfrak{g}_{\text{inv}}{}^{\mathbb{C}})$ while $\tau_0(Z_{8z\bar{z}}) = -Z_{8z\bar{z}}$, $\tau_0(\mathcal{Z}_{8z\bar{z}}) = -\mathcal{Z}_{8z\bar{z}}$ and $\tau_0(\mathcal{K}_{z\bar{z}}) = -\mathcal{K}_{z\bar{z}}$, a proper basis of the invariant real form is, in terms of $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$ generators: $\mathfrak{g}_{\text{inv}} = \mathrm{Span}\{2/3(K_{88} + 2K_{z\bar{z}})\,;\, iZ_{8z\bar{z}}\,;\, i\mathcal{Z}_{8z\bar{z}}\} \oplus \mathrm{Span}\{2(K_{88} - K_{z\bar{z}})\} \oplus \mathrm{Span}\{-i\mathcal{K}_{z\bar{z}}\}$. From now on, the last two abelian factors will be replaced by $H^{[2]}$ and

Q_A	generators
0	$K_{z\bar{z}} = -\frac{1}{6}(2H_6 + H_7) + \frac{1}{2}H_8$ $K_{88} = \frac{1}{3}(2H_6 + H_7) + \frac{1}{2}H_8$ $Z_{8z\bar{z}} = \frac{i}{2}(E_8 + F_8)\,, \quad \mathcal{Z}_{8z\bar{z}} = i(E_8 - F_8)$ $\mathcal{K}_{z\bar{z}} = i(E_7 - F_7)$
± 1	$\left\{ \begin{matrix} K_{8\bar{z}} \\ K_{8z} \end{matrix} \right\} = \frac{1}{2\sqrt{2}}\left(E_6 + F_6 \pm i(E_{67} + F_{67})\right)$ $\left\{ \begin{matrix} \mathcal{K}_{8\bar{z}} \\ \mathcal{K}_{8z} \end{matrix} \right\} = \frac{1}{\sqrt{2}}\left(E_6 - F_6 \pm i(E_{67} - F_{67})\right)$
± 2	$\left\{ \begin{matrix} K_{\bar{z}\bar{z}} \\ K_{zz} \end{matrix} \right\} = \frac{1}{2}(H_7 \pm i(E_7 + F_7))$

Table 2: Algebraic charges for $S^1 \times T^2/\mathbb{Z}_{n>2}$ orbifolds

$i\widetilde{H}^{[2]}$. Now, how such a basis behaves under the associated Cartan involution ϕ is clear from Section 4.2. This determines the invariant real form to be $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(1,1) \oplus \mathfrak{u}(1)$, with total signature $\sigma = 1$. In general, the signature of the subalgebra kept invariant by a $T^{2n}/\mathbb{Z}_{n>2}$ orbifold will be given by $\sigma(\mathfrak{g}^U) - 2n$ (keeping in mind that some orbifolds are equivalent under a trivial 2π rotation).

The coset defining the non-linear realization of the orbifolded supergravity is obtained in the usual way by modding out by the maximal compact subgroup:

$$\frac{SL(2,\mathbb{R})}{SU(2)} \times \frac{SO(1,1)}{\mathbb{Z}_2} \ .$$

In $D = 7$, there appears an additional simple root α_5, which, in the purely toroidal compactification, enhances and reconnects the U-duality algebra into $\mathfrak{g}^U = \mathfrak{sl}(5,\mathbb{R})$, following the well known $\mathfrak{e}_{n|n}$ serie. The complexification $(\mathfrak{g}^U)^{\mathbb{C}}$ resulting from orbifolding the theory calls for six additional generators: $\{K_{7\bar{z}}, Z_{7z\bar{z}}, \mathcal{K}_{7\bar{z}}, \mathcal{Z}_{7z\bar{z}}\}$ and the 2 corresponding Hermitian conjugates, produced by acting with $\mathrm{ad}(E_5 \pm F_5)$ on the objects in Table 2, all of which, together with the Cartan element K_{77}, have the expected orbifold charges.

Beside these natural combinations, we now have four new types of objects with charge ± 1, namely:

$$\left\{ \begin{matrix} Z_{78\bar{z}} \,/\, \frac{1}{2}\mathcal{Z}_{78\bar{z}} \\ Z_{78z} \,/\, \frac{1}{2}\mathcal{Z}_{78z} \end{matrix} \right\} = \frac{1}{2\sqrt{2}}\left(E_{5678}(+/-)F_{5678} \pm i(E_{568}(+/-)F_{568})\right) \ , \tag{5.2}$$

so that the invariant subalgebra $(\mathfrak{g}_{\mathrm{inv}})^{\mathbb{C}}$ is a straightforward extension by α_5 of the $D = 8$ case, as can be seen in Table 3. Its real form is obtained from the sequence (4.10) just as in $D = 8$, yielding the expected $\mathfrak{g}_{\mathrm{inv}} = \mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{so}(1,1) \oplus \mathfrak{u}(1)$, where the non-compact abelian factor is now generated by the combination

$$H^{[2]} = 4H_5 + 6H_6 + 3H_7 + 2H_8 = \frac{10}{3}\left(K_7{}^7 + K_8{}^8 - K_z{}^z\right) , \tag{5.3}$$

while, as before, $\mathfrak{u}(1) = \mathbb{R}(E_7 - F_7)$. Thus $\sigma(\mathfrak{g}_{\rm inv}) = 2$, while the total rank is again conserved by the orbifold projection.

The above procedure can be carried out in $D = 6$. In this case however, the invariant combination $H^{[2]}$ which generated earlier the non-compact $\mathfrak{so}(1,1)$ factor is now dual to a root of $\mathfrak{g}^U = \mathfrak{so}(5,5)$, namely:

$$H^{[2]} = H_{\theta_{D_5}} = \frac{2}{3}\left(K_6{}^6 + K_7{}^7 + K_8{}^8 - K_z{}^z\right) . \tag{5.4}$$

The abelian factor is thus enhanced to a full $\mathfrak{sl}(2,\mathbb{R})$ subalgebra with root system $\{\pm\theta_{D_5}\}$, while the real invariant subalgebra clearly becomes $\mathfrak{sl}(4,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{u}(1)$. In $D = 5$, θ_{D_5} connects to α_3 giving rise to $\mathfrak{g}_{\rm inv} = \mathfrak{sl}(6,\mathbb{R}) \oplus \mathfrak{u}(1)$. The extension to $D = 4,3$ is completely straightforward, yielding respectively $\mathfrak{g}_{\rm inv} = \mathfrak{so}(6,6) \oplus \mathfrak{u}(1)$ and $\mathfrak{e}_{7|7} \oplus \mathfrak{u}(1)$. The whole serie of real invariant subalgebras appears in Table 3, beside their Satake diagram, which encodes the set of simple invariant roots Π_0 and the Cartan involution ϕ.

5.1 Affine central product and the invariant subalgebra in $D = 2$

New algebraic features appear in $D = 2$, since, in the purely toroidal case, the U-duality algebra is now conjectured to be the affine $\mathfrak{e}_{9|10} \doteq \text{Split}(\hat{\mathfrak{e}}_8)$.[8]

The invariant subalgebra $(\mathfrak{g}_{\rm inv})^{\mathbb{C}}$ consists in the affine $\hat{\mathfrak{e}}_7$ together with the Heisenberg algebra $\hat{\mathfrak{u}}(1)^{\mathbb{C}}$ spanned by $\{z^n \otimes (E_7 - F_7), \forall n \in \mathbb{Z}\,; c\,; d\}$. Though both terms commute at the level of loop-algebras, their affine extensions share the same central charge $c = H_\delta$ and scaling operator d. Now, a product of two finite-dimensional algebras possessing (at least partially) a common centre is called a *central product* in the mathematical literature. The present situation is a natural generalization of this construction to the infinite-dimensional setting, where not only the central charge but also the scaling element are in common. Since the latter is a normalizer, we are not strictly dealing with a central product. We will therefore refer to such an operation as an *affine central product* and denote it by the symbol $\bowtie$. Anticipating the very-extended $D = 1$ case, we can write the invariant subalgebra as the complexification

$$\hat{\mathfrak{e}}_7 \bowtie \hat{\mathfrak{u}}(1) \equiv \left(\hat{\mathfrak{e}}_7 \oplus \hat{\mathfrak{u}}(1)\right) / \{\mathfrak{z}, \bar{d}\} , \tag{5.5}$$

where $\mathfrak{z} = H_{\delta_{E_7}} - c_{\hat{\mathfrak{u}}(1)}$ is the centre of the algebra and $\bar{d} = d_{\delta_{E_7}} - d_{\hat{\mathfrak{u}}(1)}$ is the difference of scaling operators.

The real form $\mathfrak{g}_{\rm inv}$ can be determined first by observing that the non-compact and compact generators H_α and $E_\alpha \pm F_\alpha$ (with $(\alpha|\alpha_7) = 0$) of the $\mathfrak{e}_{7|7}$ factor in $D = 3$ naturally extend to the $(t^n \pm t^{-n}) \otimes H_\alpha$ and $(t^n \otimes E_\alpha \pm t^{-n} \otimes F_\alpha)$ vertex operators of an affine $\hat{\mathfrak{e}}_{7|9}$ and second, by noting that the remaining factor in the central product $\hat{\mathfrak{e}}_7 \bowtie \hat{\mathfrak{u}}(1)$ is in fact

[8] Since $\vartheta_C(\delta) = -\delta$ implies $\vartheta_C(d) = -d$ and $\vartheta_C(c) = -c$, the split form of any KMA has signature $\sigma = \dim \mathfrak{h}$.

the loop algebra $\mathcal{L}(\mathfrak{u}(1))$ whose tower of generators can be grouped in pairs of one compact and one non-compact generator, according to [9]

$$\phi\big((z^n \pm z^{-n}) \otimes (E_7 - F_7)\big) = \pm(z^n \pm z^{-n}) \otimes (E_7 - F_7)\,, \tag{5.6}$$

in addition to the former compact Cartan generator $i\widetilde{H}^{[2]} = E_7 - F_7$. In short, the $\mathcal{L}(\mathfrak{u}(1))$ factor contributes -1 to the signature of $\mathfrak{g}_{\rm inv}$, so that in total: $\sigma = 8$. Restoring the central charge and the scaling operator in $\mathcal{L}(\mathfrak{u}(1))$ so as to write $\mathfrak{g}_{\rm inv}$ in the form (5.5), we will denote the resulting real Heisenberg algebra $\hat{\mathfrak{u}}_{|1}(1)$, so as to render its signature apparent.

For the sake of clarity, we will represent $\mathfrak{g}_{\rm inv}$ in Table 3 by the Dynkin diagram of $\hat{\mathfrak{e}}_{7|9} \oplus \hat{\mathfrak{u}}_{|1}(1)$ supplemented by the signature $\sigma(\mathfrak{g}_{\rm inv})$, but it should be kept in mind that $\mathfrak{g}_{\rm inv}$ is really given by the quotient (5.5). In Table 3, we have separated the $D = 2, 1$ cases from the rest, to stress that the Satake diagram of (Π_0, ϕ) describes $\mathfrak{g}_{\rm inv}$ completely only in the finite case.

Finally, to get some insight into the structure of the algebra $\hat{\mathfrak{e}}_7 \bowtie \hat{\mathfrak{u}}(1)$, it is worthwhile noting that the null root of the original $\mathfrak{e}_9$ is also the null root of $\hat{\mathfrak{e}}_7$, as

$$\begin{aligned}\delta &= \alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8 \\ &= \alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_8 + 2\theta_{D_5} = \delta_{E_7}\ .\end{aligned}$$

Although the root space $\mathfrak{g}_\delta \subset \mathfrak{e}_9$ is eight-dimensional, we have $\mathrm{mult}(\delta_{E_7}) = 7$, since the eighth generator $z \otimes H_7$ of $\mathfrak{g}_\delta$ is projected out. The latter is now replaced by the invariant combination $z \otimes (E_7 - F_7)$ whose commutator with itself creates the central charge of the $\hat{\mathfrak{u}}(1)$, whereas the seven remaining invariant generators $\{z \otimes H_{\theta_{D_5}}\, ; z \otimes H_i, \forall i = 1, \cdots, 5, 8\}$ build up the root space $\mathfrak{g}_{\delta_{E_7}}$. In a sense that will become clearer in $D = 1$, the multiplicity of δ_{E_9} is thus preserved in $\hat{\mathfrak{e}}_7 \bowtie \hat{\mathfrak{u}}(1)$.

5.2 A Borcherds symmetry of orbifolded M-theory in $D = 1$

In $D = 1$, finally, plenty of new $\mathfrak{sl}(10, \mathbb{R})$-tensors appear as roots of $\mathfrak{e}_{10}$, so it is now far from obvious whether the invariant subalgebra constructed from $\hat{\mathfrak{e}}_7 \bowtie \hat{\mathfrak{u}}(1)$ by adding the node α_{-1} exhausts all invariant objects. Moreover, the structure of such an algebra, as well as its Dynkin diagram is not a priori clear, since we know of no standard way to reconnect the two factors of the central product through the extended node α_{-1}. As a matter of fact, mathematicians are aware that invariant subalgebras of KMA under finite-order automorphisms might not be KMA, but can be Borcherds algebras or EALA [67, 38, 39]. Despite these preliminary reservations, we will show that the real invariant subalgebra $\mathfrak{g}_{\rm inv}$ in $D = 1$ can nevertheless be described in a closed form by a Satake diagram and the Conjecture 5.1 below, while its root system and root multiplicities can in principle be determined up to arbitrary height by a proper level decomposition.

Conjecture 5.1 *The invariant subalgebra of $\mathfrak{e}_{10}$ under the automorphism $\mathcal{U}_2^{\mathbb{Z}_n}$ is the direct sum of a $\mathfrak{u}(1)$ factor and a Borcherds algebra with degenerate Cartan matrix characterized*

[9]Note that the combination used in eqn. (5.6) is well-defined in $\mathfrak{e}_{9|10}$, since it can be rewritten in the following form: $(z^n \otimes E_7 \mp z^{-n} \otimes F_7) \pm (z^{-n} \otimes E_7 \mp z^n \otimes F_7)$.

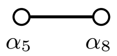
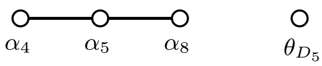
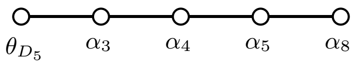
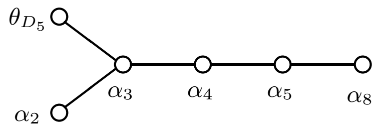
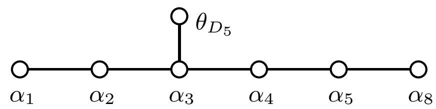
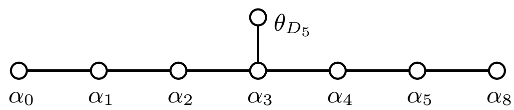
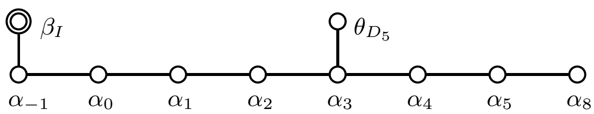
D	(Π_0, ϕ)	$\mathfrak{g}_{\text{inv}}$	$\sigma(\mathfrak{g}_{\text{inv}})$
8	 $\times$ $H^{[2]}$ $\times$ $i\widetilde{H}^{[2]}$	$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(1,1) \oplus \mathfrak{u}(1)$	1
7	 $\times$ $H^{[2]}$ $\times$ $i\widetilde{H}^{[2]}$	$\mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{so}(1,1) \oplus \mathfrak{u}(1)$	2
6	 $\times$ $i\widetilde{H}^{[2]}$	$\mathfrak{sl}(4,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{u}(1)$	3
5	 $\times$ $i\widetilde{H}^{[2]}$	$\mathfrak{sl}(6,\mathbb{R}) \oplus \mathfrak{u}(1)$	4
4	 $\times$ $i\widetilde{H}^{[2]}$	$\mathfrak{so}(6,6) \oplus \mathfrak{u}(1)$	5
3	 $\times$ $i\widetilde{H}^{[2]}$	$\mathfrak{e}_{7\|7} \oplus \mathfrak{u}(1)$	6
2	 $\times$ $\left\{i\widetilde{H}_n^{[2]}\right\}_{n\in\mathbb{Z}}$	$\hat{\mathfrak{e}}_{7\|9} \oplus \mathcal{L}(\mathfrak{u}(1))_{\|-1}$	8
1	 $\times$ $i\widetilde{H}^{[2]}$	${}^2\mathcal{B}_{10\|11} \oplus \mathfrak{u}(1)$	8

Table 3: The split subalgebras $\mathfrak{g}_{\text{inv}}$ for $T^{9-D} \times T^2/\mathbb{Z}_{n>2}$ compactifications.

by one isotropic imaginary simple root β_I of multiplicity one and nine real simple roots, modded out by its centre and its derivation.

As already mentioned in Section 5.1, we choose to represent, in Table 3, the real form $\mathfrak{g}_{\text{inv}}$ before quotientation, by the Dynkin diagram of its defining Borcherds algebra ${}^2\mathcal{B}_{10}$. Both are related through

$$\mathfrak{g}_{\text{inv}} = \mathfrak{u}(1) \oplus {}^2\mathcal{B}_{10|11} \,/\, \{\mathfrak{z}, d_I\}\,, \tag{5.7}$$

which is an extension of the affine central product (5.5) encountered in $D = 2$, provided we now set $\mathfrak{z} = H_\delta - H_I$. We also define $H_I \doteq H_{\beta_I}$ and $d_I \doteq d_{\beta_I}$ as the Cartan generator dual to β_I and the derivation counting levels in β_I.

More precisely, ${}^2\mathcal{B}_{10}$ has the following 10×10 degenerate Cartan matrix, with rank $r = 9$

$$A = \left(\begin{array}{ccc|c} 0 & -1 & 0 & \mathbb{O} \\ -1 & 2 & -1 & \\ \hline 0 & -1 & & \\ \mathbb{O} & & & A(\hat{\mathfrak{e}}_7) \end{array} \right) ,$$

and it can be checked that its null vector is indeed the centre $\mathfrak{z}$ of the Borcherds algebra mentioned above. As for affine KMA, the Cartan subalgebra of Borcherds algebras with a non-maximal $n \times n$ Cartan matrix has to be supplemented by $n - r$ new elements that allow to discriminate between roots having equal weight under $\mathrm{Ad}(H_i)$, $\forall i = 1, .., n$. Here, the Cartan subalgebra of ${}^2\mathcal{B}_{10}$ thus contains a derivation d_I counting the level in β_I, allowing, for example, to distinguish between $2\beta_I$,[10] $\beta_I + \delta$ and 2δ, which all have weights -2 under H_{-1} and 0 under all other Cartan generators dual to simple roots. However, the operator d_I is not in $\mathfrak{e}_{10}$ and consequently not in $\mathfrak{g}_{\mathrm{inv}}$, either. Hence, the quotient by $\{\mathfrak{z}, \bar{d}\}$ in Conjecture 5.1 amounts to identifying H_I with H_δ. Furthermore, since the roots α_{-1} and β_I are connected on the Dynkin diagram, $-H_{-1}$ plays, already in ${}^2\mathcal{B}_{10}$, the same rôle as d_I with respect to β_I. So the elimination of d_I by the quotient (5.7) is equivalent to identifying it with $-H_{-1}$, which parallels the treatment of H_I with H_δ.

These two processes reconstruct in $\mathfrak{g}_{\mathrm{inv}}$ the 8-dimensional root space $(\mathfrak{g}_{\mathrm{inv}})_\delta = (\mathfrak{g}_{{}^2\mathcal{B}_{10}})_\delta \oplus (\mathfrak{g}_{{}^2\mathcal{B}_{10}})_{\beta_I}$ inherited from $\mathfrak{e}_{10}$.

Formally, one decomposes:

$$E^a_\delta|_{\mathfrak{g}_{\mathrm{inv}}} = E^a_\delta|_{{}^2\mathcal{B}_{10}}\,, \forall a = 1, \dots, 7\,, \quad \text{and} \quad E^8_\delta|_{\mathfrak{g}_{\mathrm{inv}}} = E_{\beta_I} \doteq \frac{1}{\sqrt{2}}\left(E_{\delta+\alpha_7} - E_{\delta-\alpha_7}\right),$$

and $F_{\beta_I} = (E_{\beta_I})^\dagger$ in $\mathfrak{e}_{10}$. One should thus pay attention to the fact that although $\beta_I \sim \delta$ in $\mathfrak{g}_{\mathrm{inv}}$, their corresponding ladder operators remain distinct.

We have chosen to depict the Borcherds algebra under scrutiny by the Dynkin diagram displayed in Table 3. However such a GKMA, let alone its root multiplicities, is not known in the literature. So at this stage, one must bear in mind that the Dynkin diagram we associate to ${}^2\mathcal{B}_{10}$ is only meant to determine the correct root lattice for $(\mathfrak{g}_{\mathrm{inv}})^{\mathbb{C}}$. The root multiplicities, on the other hand, have to be computed separately by decomposing root-spaces of $\mathfrak{e}_{10}$ into root-spaces of $(\mathfrak{g}_{\mathrm{inv}})^{\mathbb{C}}$. So we need *both* the Dynkin diagram of ${}^2\mathcal{B}_{10}$ *and* the root multiplicities listed in Table 7 in order to determine $\mathfrak{g}_{\mathrm{inv}}$ completely.

In order to support the conjecture 5.1, we start by performing a careful level by level analysis. We proceed by decomposing $\mathfrak{e}_{10}$ with respect to the coefficient of α_8 into tensorial irreducible representations of $\mathfrak{sl}(10, \mathbb{R})$. Such representations, together with the multiplicity of the weights labelling them, are summarized up to level $l = 6$ in α_8 in Tables 4 and 5.

[10] In contrast to real simple roots, we expect for isotropic simple roots of a Borcherds algebra that $n\beta_I \in \Delta$, $\forall n \in \mathbb{Z}$.

l	$\mathcal{R}(\Lambda)$	Λ	$\dim \mathcal{R}(\Lambda)$
0	$K_{(ij)}$	[200000000]	$55 = 45 + 10$ Cartan
1	$Z_{[ijk]}$	[001000000]	120
2	$\widetilde{Z}_{[i_1\cdots i_6]}$	[000001000]	210
3	$\widetilde{K}_{(i)[j_1\cdots j_8]}$	[100000010]	$440 = 360 + 8 \times 10_{[0]}$
4	$(\widetilde{K}_{(i)} \otimes Z)_{[j_1\cdots j_8][k_1k_2k_3]}$	[001000001]	$1155 = 840 + 7 \times 45_{[0]}$
	$A_{(ij)}$	[200000000]	$55 = 10 + 45_{[0]}$
5	$(\widetilde{K}_{(i)} \otimes \widetilde{Z})_{[j_1\cdots j_8][k_1\cdots k_6]}$	[000001001]	$1848 = 840 + 4 \times 252_{[0]}$
	$B_{(i)[j_1\cdots j_4]}$	[100100000]	$1848 = 840 + 4 \times 252_{[0]}$
6	$(\widetilde{K}_{(i)} \otimes \widetilde{K}_{(j)})_{[k_1\cdots k_8][l_1\cdots l_8]}$	[100000011]	$3200 = 720 + 2 \times 840 + 16 \times 45 + 8 \times 10_{[0]}$
	$(A \otimes \widetilde{Z})_{(ij)[k_1\cdots k_6]}$	[010001000]	$8250 = 3150 + 5 \times 840 + 20 \times 45$
	$D_{(i)[j_1\cdots j_7]}$	[100000100]	$1155 = 840 + 7 \times 45$
	$S_{[i_1\cdots i_8]}$	[000000010]	45

Table 4: Representations of $\mathfrak{sl}(10,\mathbb{R})$ in $\mathfrak{e}_{10}$ up to $l = 6$

These tables have been deduced from the low-level decomposition of roots of $\mathfrak{e}_{10}$ that can be found up to level 18 in [52]. Since we are more interested in the roots themselves and their multiplicities than in the dimension of the corresponding $\mathfrak{sl}(10,\mathbb{R})$ representations, we added, in column $\dim \mathcal{R}(\Lambda)$ of Table 4, the way the dimension of each representation decomposes in generators corresponding to different sets of roots, obtained by all reflections by the Weyl group of $\mathfrak{sl}(10,\mathbb{R})$ (i.e. permutations of indices in the physical basis) on the highest weight and possibly other roots. In the first column of Table 5, the tensor associated to the highest weight is defined, the highest weight being obtained by setting all indices to their maximal values.

Note that roots that are permutations of the highest weight of no representation, or in other words, have null outer multiplicity, do not appear in Table 4, contrary to what is done in [52]. However, these can be found in Table 5. The order of the orbits under the Weyl group of $\mathfrak{sl}(10,\mathbb{R})$ is given in column O_w^{10}, in which representations of null outer multiplicity are designated by a [0] subscript. Besides, column m contains the root multiplicities, while column $|\Lambda|^2$ contains the squared length, which, in the particular case of $\mathfrak{e}_{10}$, provide equivalent characterizations.

For example, the representation with Dynkin labels [000001000] at level 3 is composed of the Weyl orbit of its highest weight generator $\widetilde{K}_{(10)[3\cdots 10]}$ together with 8 Weyl orbits of the (outer multiplicity 0) root $\widetilde{K}_{(2)[3\cdots 10]}$ for a total size $360 + 8 \times 10$. Similarly, the representation [001000001] at level 4 is composed of the 840 components of the associated

Generator	α	Physical basis	O_w^{10}	$m(\alpha)$	$\lvert\alpha\rvert^2$
$K_{9\,10}$	0 0 0 0 0 0 0 0 0 1	$(0,0,0,0,0,0,0,0,1,-1)$	45	1	2
$Z_{[89\,10]}$	1 0 0 0 0 0 0 0 0 0	$(0,0,0,0,0,0,0,1,1,1)$	120	1	2
$\widetilde{Z}_{[56789\,10]}$	2 0 0 0 0 1 2 3 2 1	$(0,0,0,0,1,1,1,1,1,1)$	210	1	2
$\widetilde{K}_{(10)[3\cdots10]}$	3 0 0 1 2 3 4 5 3 1	$(0,0,1,1,1,1,1,1,1,2)$	360	1	2
$\widetilde{K}_{(2)[3\cdots10]}$	3 0 1 2 3 4 5 6 4 2	$(0,1,1,1,1,1,1,1,1,1)$	$10_{[0]}$	8	0
$(\widetilde{K}_{(2)}\otimes Z)_{[3\cdots10][89\,10]}$	4 0 1 2 3 4 5 6 4 2	$(0,1,1,1,1,1,1,2,2,2)$	840	1	2
$A_{(10\,10)}$	4 1 2 3 4 5 6 7 4 1	$(1,1,1,1,1,1,1,1,1,3)$	10	1	2
$(\widetilde{K}_{(2)}\otimes Z)_{[3\cdots10][19\,10]}$ $A_{(9\,10)}$	4 1 2 3 4 5 6 7 4 2	$(1,1,1,1,1,1,1,1,2,2)$	$45_{[0]}$	8	0
$(\widetilde{K}_{(2)}\otimes\widetilde{Z})_{[3\cdots10][5\cdots10]}$	5 0 1 2 3 5 7 9 6 3	$(0,1,1,1,2,2,2,2,2,2)$	840	1	2
$B_{(10)[7\cdots10]}$	5 1 2 3 4 5 6 8 5 2	$(1,1,1,1,1,1,2,2,2,3)$	840	1	2
$(\widetilde{K}_{(2)}\otimes\widetilde{Z})_{[3\cdots10][16\cdots10]}$ $B_{(6)[7\cdots10]}$	5 1 2 3 4 5 7 9 6 3	$(1,1,1,1,1,2,2,2,2,2)$	$252_{[0]}$	8	0
$(\widetilde{K}_{(2)}\otimes\widetilde{K}_{(10)})_{[3\cdots10][3\cdots10]}$	6 0 1 3 5 7 9 11 7 3	$(0,1,2,2,2,2,2,2,2,3)$	720	1	2
$(A\otimes\widetilde{Z})_{(9\,10)[5\cdots10]}$	6 1 2 3 4 6 8 10 6 3	$(1,1,1,1,2,2,2,2,3,3)$	3150	1	2
$(\widetilde{K}_{(1)}\otimes\widetilde{K}_{(10)})_{[24\cdots10][3\cdots10]}$ $(A\otimes\widetilde{Z})_{[4\,10][5\cdots10]}$ $D_{(10)[4\cdots10]}$	6 1 2 3 5 7 9 11 7 3	$(1,1,1,2,2,2,2,2,2,3)$	840	8	0
$(\widetilde{K}_{(2)}\otimes\widetilde{K}_{(2)})_{[3\cdots10][3\cdots10]}$	6 0 2 4 6 8 10 12 8 4	$(0,2,2,2,2,2,2,2,2,2)$	$10_{[0]}$	8	0
$(\widetilde{K}_{(1)}\otimes\widetilde{K}_{(10)})_{[2\cdots9][3\cdots10]}$ $(A\otimes\widetilde{Z})_{(34)[5\cdots10]}$ $D_{(3)[4\cdots10]}$ $(\widetilde{K}_{(1)}\otimes\widetilde{K}_{(2)})_{[3\cdots10][3\cdots10]}$ $S_{[3\cdots10]}$	6 1 2 4 6 8 10 12 8 4	$(1,1,2,2,2,2,2,2,2,2)$	45	44	-2

Table 5: Decomposition of root spaces of $\mathfrak{e}_{10}$ into $\mathfrak{sl}(10,\mathbb{R})$ representations

tensor, together with 7 copies of the anti-symmetric part of A_{ij} that corresponds to a root of multiplicity 8 and outer multiplicity 0, for a total dimension $840+7\times 45$. The remaining eighth copy combines with the (inner and outer) multiplicity 1 diagonal part $A_{(ii)}$ to form a symmetric tensor [200000000]. Note that the $A_{(ij)}$ representation differs from the $K_{(ij)}$ one, first because the diagonal elements of the latter are given by Cartan elements and not ladder operators as in $A_{(ij)}$, and second because these two representations obviously correspond to roots of totally different level, height and threshold. Clearly, isomorphic irreducible representations of $\mathfrak{sl}(10,\mathbb{R})$ can appear several times in the decomposition of $\mathfrak{e}_{10}$.

Moreover, and more interestingly, weights with different physical interpretations may live in the same representation of $\mathfrak{e}_{10}$. In particular, the third weight $\widetilde{K}_{(10)[3\cdots 10]}$ in Table 5 is clearly related to the corresponding Euclidean Kaluza-Klein monopole (KK7M), while the fourth weight $\widetilde{K}_{(2)[3\cdots 10]}$, though belonging to the same [100000010] representation, corresponds, according to the proposal of [21] (cf. Table 1) to the Minkowskian Kaluza-Klein particle (KKp) G_{01}. Similarly, the seventh weight $A_{(10\,10)}$ is associated to the conjectured Euclidean KK9M-brane $W_{(10\,10)[1\cdots 10]}$, while $A_{(9\,10)}$ is interpreted as the Minkowskian M2-brane $C_{09\,10}$. To complete the list of Minkowskian objects, we have in addition the weights $B_{(6)[7\cdots 10]}$ and $D_{(10)[4\cdots 10]}$ related respectively to the exceptional M5-brane $\widetilde{C}_{06\cdots 10}$ and the Kaluza-Klein monopole (KK7M) $\widetilde{G}_{(10)04\cdots 10}$.

After this short excursion into weights and representations of $\mathfrak{e}_{10}$, let us come back to the characterization of ${}^2\mathcal{B}_{10}$. Observing that objects commuting with $i\widetilde{H}^{[2]} = -i\mathcal{K}_{z\bar{z}}$ have the form $X_{\cdots(99)} - X_{\cdots(10\,10)}$ or $X_{\cdots[9\,10]}$, we are naturally looking for invariant combinations of generators of $\mathfrak{e}_{10}$ with such tensorial properties. The latter can then be decomposed into $\mathfrak{sl}(8,\mathbb{R})$ tensors with Weyl orbits of order O^8_w and identified with a root of ${}^2\mathcal{B}_{10}$. We have carried out such a decomposition up to $l=6$ in α_8 and listed the corresponding roots of ${}^2\mathcal{B}_{10}$, together with their multiplicities m, in Table 7.

In order to make clear how to retrieve the root system of $\mathfrak{g}_{\text{inv}}$ from Table 7, we give in the third column the expression of a given root of ${}^2\mathcal{B}_{10}$ in a generalized notation for the physical basis, denoted *physical eigenbasis of* $\mathfrak{e}_{10}$. This eigenspace basis is defined by[11]:

$$E'_i = E_i\,,\ \forall i=-1,\ldots,5,8\,,\quad E'_6 = \frac{1}{\sqrt{2}}(E_6 + iE_{67})\,,\quad E'_7 = \frac{1}{2}(H_7 - i(E_7+F_7))\,, \quad (5.8)$$

so that

$$[E_{\alpha'}, E'_7/F'_7] = \mp i(E_{\alpha+\alpha_7} - E_{\alpha-\alpha_7})\,, \quad (5.9)$$

for all α''s satisfying $|\alpha'|^2 \leq 0$ and $\alpha' = \alpha$, where α is a root of $\mathfrak{e}_{10}$ in the original basis. In fact, all invariant generators in $\mathfrak{e}_{10}$ either satisfy $E_{\alpha'} = E_\alpha$, or are of the form (5.9). In Table 7, we characterize the former by their root $\alpha' = \alpha$ in the physical eigenbasis, and the latter as the sum of a root $\alpha' = \alpha$ and $-\alpha'_7$, to emphasize the fact that they build separate root spaces of ${}^2\mathcal{B}_{10}$ that will merge in $\mathfrak{g}_{\text{inv}}$. Indeed, modding out ${}^2\mathcal{B}_{10}$ by $\{\mathfrak{z};\bar{d}\}$ eliminates the Cartan elements measuring the level in $\beta_I = \delta - \alpha'_7$ in $\mathfrak{g}_{\text{inv}}$, thus identifying β_I with δ.

[11] This basis will be used again for computing shift vectors in Section 9.

l	Invariant Tensor	Physical eigenbasis of $\mathfrak{e}_{10}$	O_w^8	$\alpha \in \Delta_+(^2\mathcal{B}_{10})$	$m(\alpha)$
0	$K_{[78]}$	$(0,0,0,0,0,0,1,-1,0,0)'$	28	0 0 0 0 0 0 0 0 1 0	1
1	$Z_{[89\,10]}$	$(0,0,0,0,0,0,0,1,1,1)'$	8	0 0 0 0 0 0 0 0 0 1	1
	$Z_{[678]}$	$(0,0,0,0,0,1,1,1,0,0)'$	56	0 1 0 0 0 0 0 0 0 0	1
2	$\widetilde{Z}_{[56789\,10]}$	$(0,0,0,0,1,1,1,1,1,1)'$	70	0 1 0 0 0 0 1 1 1 1	1
	$\widetilde{Z}_{[345678]}$	$(0,0,1,1,1,1,1,1,0,0)'$	28	0 2 0 0 1 2 3 2 1 0	1
3	$\widetilde{K}_{(9)[2\cdots 89]} - \widetilde{K}_{(10)[2\cdots 810]}$	$(0,1,1,1,1,1,1,1,1,1)' + (0^8,-1,1)'$	8	1 0 0 0 0 0 0 0 0 0	1
	$\widetilde{K}_{(8)[3\cdots 10]}$	$(0,0,1,1,1,1,1,2,1,1)'$	168	0 2 0 0 1 2 3 2 1 1	1
	$\widetilde{K}_{(2)[3\cdots 10]}$	$(0,1,1,1,1,1,1,1,1,1)'$	8	0 2 0 1 2 3 4 3 2 1	7
	$\widetilde{K}_{(8)[1\cdots 8]}$	$(1,1,1,1,1,1,1,2,0,0)'$	8	0 3 1 2 3 4 5 3 1 0	1
4	$\big((\widetilde{K}\otimes Z)_{[1\cdots 9][789]}$ $-(\widetilde{K}\otimes Z)_{[1\cdots 8\,10][78\,10]}\big)$	$(1,1,1,1,1,1,2,2,1,1)' + (0^8,-1,1)'$	28	1 1 1 1 1 1 1 0 0 0	1
	$(\widetilde{K}_{(2)}\otimes Z)_{[3\cdots 10][89\,10]}$	$(0,1,1,1,1,1,1,2,2,2)'$	56	0 2 0 1 2 3 4 3 2 2	1
	$(\widetilde{K}_{(2)}\otimes Z)_{[3\cdots 10][678]}$	$(0,1,1,1,1,2,2,2,1,1)'$	280	0 3 0 1 2 3 4 3 2 1	1
	$(\widetilde{K}_{(2)}\otimes Z)_{[3\cdots 10][19\,10]}$ $A_{(9\,10)}$	$(1,1,1,1,1,1,1,1,2,2)'$	1	0 2 1 2 3 4 5 4 3 2	7
	$(\widetilde{K}_{(2)}\otimes Z)_{[3\cdots 10][178]}$ $A_{(78)}$	$(1,1,1,1,1,1,2,2,1,1)'$	28	0 3 1 2 3 4 5 3 2 1	7
	$A_{(9\,9)} - A_{(10\,10)}$	$(1,1,1,1,1,1,1,1,2,2)' + (0^8,-1,1)'$	1	1 0 1 1 1 1 1 1 1 1	1
	$A_{(88)}$	$(1,1,1,1,1,1,1,3,1,1)'$	8	0 3 1 2 3 4 5 3 1 1	1
5	$\big((\widetilde{K}\otimes\widetilde{Z})_{[1\cdots 9][4\cdots 9]}$ $-(\widetilde{K}\otimes\widetilde{Z})_{[1\cdots 810][4\cdots 8\,10]}\big)$	$(1,1,1,2,2,2,2,2,1,1)' + (0^8,-1,1)'$	56	1 2 1 1 1 2 3 2 1 0	1
	$(\widetilde{K}_{(2)}\otimes\widetilde{Z})_{[3\cdots 10][5\cdots 10]}$	$(0,1,1,1,2,2,2,2,2,2)'$	280	0 3 0 1 2 3 5 4 3 2	1

Table 6: Decomposition of root spaces of $^2\mathcal{B}_{10}$ in representations of $\mathfrak{sl}(8,\mathbb{R})$.

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5	$(\widetilde{K}_{(2)} \otimes \widetilde{Z})_{[3\cdots10][3\cdots\,8]}$	$(0,1,2,2,2,2,2,2,1,1)'$	56	0 4 0 1 3 5 7 5 3 1	1
	$B_{(8)[7\cdots\,10]}$	$(1,1,1,1,1,1,2,3,2,2)'$	56	0 3 1 2 3 4 5 3 2 2	1
	$B_{(8)[5\cdots\,8]}$	$(1,1,1,1,2,2,2,3,1,1)'$	280	0 4 1 2 3 4 6 4 2 1	1
	$B_{(9)[678\,9]} - B_{(10)[678\,10]}$	$(1,1,1,1,1,2,2,2,2,2)' + (0^8,-1,1)'$	56	1 1 1 1 1 1 1 1 1 1	1
	$(\widetilde{K}_{(2)} \otimes \widetilde{Z})_{[3\cdots10][16\cdots\,10]}$ $B_{(6)[7\cdots\,10]}$	$(1,1,1,1,1,2,2,2,2,2)'$	56	0 3 1 2 3 4 5 4 3 2	7
	$(\widetilde{K}_{(2)} \otimes \widetilde{Z})_{[3\cdots10][14\cdots\,8]}$ $B_{(4)[5\cdots\,8]}$	$(1,1,1,2,2,2,2,2,1,1)'$	56	0 4 1 2 3 5 7 5 3 1	7
6	$\big((\widetilde{K} \otimes \widetilde{K}_{(9)})_{[2\cdots10][2\cdots\,9]}$ $-(\widetilde{K} \otimes \widetilde{K}_{(10)})_{[2\cdots8\,10][2\cdots810]}\big)$	$(0,2,2,2,2,2,2,2,2,2)' + (0^8,-1,1)'$	8	2 0 0 0 0 0 0 0 0 0	1
	$(\widetilde{K}_{(2)} \otimes \widetilde{K}_{(8)})_{[3\cdots10][3\cdots\,10]}$	$(0,1,2,2,2,2,2,3,2,2)'$	336	0 4 0 1 3 5 7 5 3 2	1
	$\big((\widetilde{K} \otimes \widetilde{K}_{(9)})_{[1\cdots9][3\cdots\,10]}$ $-(\widetilde{K} \otimes \widetilde{K}_{(10)})_{[1\cdots8\,10][3\cdots10]}\big)$ $\big((A \otimes \widetilde{Z})_{[89][3\cdots\,79]}$ $-(A \otimes \widetilde{Z})_{[8\,10][3\cdots\,7\,10]}\big)$ $D_{(9)[3\cdots9]} - D_{(10)[3\cdots8\,10]}$	$(1,1,2,2,2,2,2,2,2,2)' + (0^8,-1,1)'$	28	2 0 1 0 0 0 0 0 0 0	8
	$(\widetilde{K}_{(2)} \otimes \widetilde{K}_{(8)})_{[3\cdots10][1\cdots\,8]}$ $(A \otimes \widetilde{Z})_{[78][2\cdots\,68]}$ $D_{(8)[2\cdots8]}$	$(1,2,2,2,2,2,2,3,1,1)'$	56	0 5 1 3 5 7 9 6 3 1	1
	$(A \otimes \widetilde{Z})_{(9\,10)[5\cdots\,10]}$	$(1,1,1,1,2,2,2,2,3,3)'$	70	0 3 1 2 3 4 6 5 4 3	1
	$(A \otimes \widetilde{Z})_{(78)[5\cdots\,10]}$	$(1,1,1,1,2,2,3,3,2,2)'$	420	0 4 1 2 3 4 6 4 3 2	1
	$(A \otimes \widetilde{Z})_{(78)[3\cdots\,8]}$	$(1,1,2,2,2,2,3,3,1,1)'$	420	0 5 1 2 4 6 8 5 3 1	1
	$\big((A \otimes \widetilde{Z})_{(89)[4\cdots\,89]}$ $-(A \otimes \widetilde{Z})_{(8\,10)[4\cdots8\,10]}\big)$	$(1,1,1,2,2,2,2,3,2,2)' + (0^8,-1,1)'$	280	1 2 1 1 1 2 3 2 1 1	1
	$\big((\widetilde{K} \otimes \widetilde{K}_{(1)})_{[1\cdots9][2\cdots\,9]}$ $-(\widetilde{K} \otimes \widetilde{K}_{(1)})_{[1\cdots8\,10][2\cdots8\,10]}\big)$	$(2,2,2,2,2,2,2,2,1,1)' + (0^8,-1,1)'$	8	2 1 2 2 2 2 2 1 0 0	8
	$(\widetilde{K}_{(2)} \otimes \widetilde{K}_{(2)})_{[3\cdots10][3\cdots\,10]}$	$(0,2,2,2,2,2,2,2,2,2)'$	8	0 4 0 2 4 6 8 6 4 2	7

Table 7: Decomposition of root spaces of ${}^2\mathcal{B}_{10}$ in representations of $\mathfrak{sl}(8,\mathbb{R})$

6	$(\widetilde{K}\otimes\widetilde{K}_{(10)})_{[1\cdots 9][3\cdots 10]}$ $(A\otimes\widetilde{Z})_{(34)[5\cdots 10]}$ $D_{(3)[4\cdots 10]}$ $(\widetilde{K}\otimes\widetilde{K}_{(2)})_{[13\cdots 10][3\cdots 10]}$ $S_{[3\cdots 10]}$	$(1,1,2,2,2,2,2,2,2,2)'$	28	0 4 1 2 4 6 8 6 4 2	36
	$(\widetilde{K}\otimes\widetilde{K}_{(8)})_{[1\cdots 79\,10][1\cdots 8]}$ $(A\otimes\widetilde{Z})_{(12)[3\cdots 8]}$ $D_{(1)[2\cdots 8]}$ $(\widetilde{K}\otimes\widetilde{K}_{(1)})_{[1\cdots 9][2\cdots 810]}$ $S_{[1\cdots 8]}$	$(2,2,2,2,2,2,2,2,1,1)'$	1	0 5 2 4 6 8 10 7 4 1	36

Table 8: Decomposition of root spaces of ${}^2\mathcal{B}_{10}$ in representations of $\mathfrak{sl}(8,\mathbb{R})$

As an example, consider the fourth and sixth root at $l = 4$ in Table 7. Both are identified in $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$:

$$(1,1,1,1,1,1,1,1,2,2)' + (0^8,-1,1)' \sim (1,1,1,1,1,1,1,1,2,2)'$$

so that their respective generators: $A_{99} - A_{10\,10}$ on the one hand, and $A_{9\,10}$ plus 6 combinations of operators of the form $\widetilde{K}_{[1\cdots\hat{i}\cdots 89\,10]} \otimes Z_{[i9\,10]}$, $i = 1,\ldots,6$ on the other hand, are now collected in a common 8-dimensional root space $(\mathfrak{g}_{\text{inv}})_\alpha$ for $\alpha = \delta+\alpha_{-1}+\ldots+\alpha_5+\alpha_8$. As a result, the root multiplicity of $\delta+\alpha_{-1}+\ldots+\alpha_5+\alpha_8$ is conserved when reducing $\mathfrak{e}_{10}$ to $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$, even though its corresponding root space is spanned by (partly) different generators in each case. We expect this mechanism to occur for all imaginary roots of $\mathfrak{g}_{\text{inv}}$.

On the other hand, the multiplicity of isotropic roots in ${}^2\mathcal{B}_{10}$ splits according to $8 \to 1+7$, in which the root space of multiplicity one is of the form (5.9). Likewise, imaginary roots of $\mathfrak{e}_{10}$ of length -2 split in ${}^2\mathcal{B}_{10}$ as $44 \to 8+36$, and we expect, though we did not push the analysis that far, that imaginary roots of length -4 will split as $192 \to 44+148$. Generally, we predict root multiplicities of ${}^2\mathcal{B}_{10}$ to be 1, 7, 36, 148, 535, 1745,... Although not our initial purpose, the method can thus be exploited to predict root multiplicities of certain Borcherds algebras constructed as fixed-point algebras of KMAs under a finite-order automorphism of order bigger than 2.

Finally, a remark on the real $\mathfrak{g}_{\text{inv}}$. As anticipated in eqn. (5.7), the Borcherds algebra involved is actually the split form ${}^2\mathcal{B}_{10|11}$. Its reality properties can be inferred from the affine case, which has been worked out in detail in section 5.1, and the behaviour of the generators $E_{n\beta_I} \doteq (1/\sqrt{2})(E_{n\delta+\alpha_7} - E_{n\delta-\alpha_7})$ and $F_{n\beta_I} = (E_{n\beta_I})^\dagger$ under the restriction ϕ.

Since $\phi(E_{n\beta_I}) = -F_{n\beta_I}$ and $\phi(F_{n\beta_I}) = -E_{n\beta_I}$, both sets of operators combine symmetrically in the usual compact and non-compact operators $E_{n\beta_I} \mp F_{n\beta_I}$. Moreover, the Cartan generator H_I has to match the reality property of H_δ to which it gets identified under relation (5.5), and must therefore be non-compact in ${}^2\mathcal{B}_{10}$, which agrees with the definition of the split form ${}^2\mathcal{B}_{10|11}$.

As in the affine case, the signature of $\mathfrak{g}_{\text{inv}}$ remains finite and is completely determined by the reality properties of the Cartan subalgebra. Taking into account the quotient (5.5), the signature is $\sigma = 8$.

6. The orbifolds $T^4/\mathbb{Z}_{n>2}$

In this section, we will treat the slightly more involved orbifold $T^{7-D} \times T^4/Z_n$ for $n \geqslant 3$. A new feature appears in this case: the invariant subalgebras will now contain generators that are complex combinations of the original $\mathfrak{e}_{10}$ generators. If the orbifold is chosen to act on the coordinates $\{x^7, x^8, x^9, x^{10}\}$, it will only affect roots containing α_4, α_5, α_6, α_7 or α_8, and the corresponding generators. This orbifold should thus be studied first in $D = 6$ where $\mathfrak{g}^U = \mathfrak{so}(5,5)$, with the following action on the complex coordinates:

$$(z_1, \bar{z}_1) \to (e^{2\pi i/n} z_1, e^{-2\pi i/n} \bar{z}_1)\,, \qquad (z_2, \bar{z}_2) \to (e^{-2\pi i/n} z_2, e^{2\pi i/n} \bar{z}_2)\,. \tag{6.1}$$

In other words, we choose the prescription $Q_1 = +1$ and $Q_2 = -1$ to ensure $\sum_i Q_i = 0$.

The rotation operator $\mathcal{U}_4^{\mathbb{Z}_n} = \prod_{k=1}^{2} e^{-\frac{2\pi i}{n} Q_k \mathcal{K}_{z_k \bar{z}_k}}$ with the above charge prescription leaves invariant the following objects:

$$\begin{array}{l|l}
Q_A = 0 & \begin{array}{rl}
K_{66} & = \frac{1}{4}(5H_4 + 6H_5 + 4H_6 + 2H_7 + 3H_8)\,, \\
K_{z_1\bar{z}_1} & = \frac{1}{2}(K_5 + K_6) = \frac{1}{4}(H_4 + 4H_5 + 4H_6 + 2H_7 + 3H_8)\,, \\
K_{z_2\bar{z}_2} & = \frac{1}{2}(K_7 + K_8) = \frac{1}{4}(H_4 + 2H_5 + 3H_8)\,, \\
\left\{\begin{array}{c} K_{\bar{z}_1\bar{z}_2} \,/\, \frac{1}{2}\mathcal{K}_{\bar{z}_1 barz_2} \\ K_{z_1 z_2} \,/\, \frac{1}{2}\mathcal{K}_{z_1 z_2} \end{array}\right\} & = \frac{1}{4}\big(E_{56} - E_{67}(+/-)(F_{56} - F_{67}) \\
 & \qquad \pm i\big(E_6 + E_{567}(+/-)(F_6 + F_{567})\big)\big)\,, \\
Z_{6z_1\bar{z}_1} \,/\, \frac{1}{2}\mathcal{Z}_{6z_1\bar{z}_1} & = \frac{i}{2}(E_{45^26^278}(+/-)F_{45^26^278}) \\
Z_{6z_2\bar{z}_2} \,/\, \frac{1}{2}\mathcal{Z}_{6z_2\bar{z}_2} & = \frac{i}{2}(E_{458}(+/-)F_{458})\,, \\
\left\{\begin{array}{c} Z_{6\bar{z}_1\bar{z}_2} \,/\, \frac{1}{2}\mathcal{Z}_{6\bar{z}_1\bar{z}_2} \\ Z_{6z_1 z_2} \,/\, \frac{1}{2}\mathcal{Z}_{6z_1 z_2} \end{array}\right\} & = \frac{1}{4}\big(E_{45^2678} - E_{4568}(+/-)(F_{45^2678} - F_{4568}) \\
 & \qquad \pm i\big(E_{45^268} + F_{45^268}(+/-)(E_{45678} + F_{45678})\big)\big)\,. \\
\mathcal{K}_{z_1\bar{z}_1} = i(E_5 - F_5)\,, & \qquad \mathcal{K}_{z_2\bar{z}_2} = i(E_7 - F_7)\,.
\end{array}
\end{array} \tag{6.2}$$

Thus $\mathfrak{g}_{\text{inv}}$ has as before (conserved) rank 5. Note that the invariant diagonal metric elements are in fact linear combinations of the three basic Cartan generators satisfying $\alpha_5(H) = \alpha_7(H) = 0$, namely $\{2H_4 + H_5, H_5 + 2H_8, H_5 + 2H_6 + H_7\}$. Furthermore, we have various charged combinations:

$$\begin{array}{l|l}
Q_A = +1 & \begin{array}{rl}
K_{6\bar{z}_1} \,/\, \frac{1}{2}\mathcal{K}_{6\bar{z}_1} & = \frac{1}{2\sqrt{2}}\big(E_4(+/-)F_4 + i(E_{45}(+/-)F_{45})\big)\,, \\
K_{6z_2} \,/\, \frac{1}{2}\mathcal{K}_{6z_2} & = \frac{1}{2\sqrt{2}}\big(E_{456}(+/-)F_{456} - i(E_{4567}(+/-)F_{4567})\big)\,, \\
Z_{z_1\bar{z}_1 z_2} \,/\, \frac{1}{2}\mathcal{Z}_{z_1\bar{z}_1 z_2} & = \frac{1}{2\sqrt{2}}\big(E_{568}(+/-)F_{568} + i(E_{5678}(+/-)F_{5678})\big)\,, \\
Z_{\bar{z}_1 z_2\bar{z}_2} \,/\, \frac{1}{2}\mathcal{Z}_{\bar{z}_1 z_2\bar{z}_2} & = \frac{1}{2\sqrt{2}}\big(-(E_8(+/-)F_8) + i(E_{58}(+/-)F_{58})\big)\,,
\end{array}
\end{array} \tag{6.3}$$

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and their complex conjugates with $Q_A = -1$, along with:

$$
\begin{array}{c|l}
Q_A = +2 & \begin{aligned}
K_{\bar z_1 \bar z_1} \quad &= \tfrac{1}{2}\big(H_5 + i(E_5 + F_5)\big)\,, \\
K_{z_2 z_2} \quad &= \tfrac{1}{2}\big(H_7 - i(E_7 + F_7)\big)\,, \\
K_{\bar z_1 z_2} \,/\, \tfrac{1}{2}\mathcal{K}_{\bar z_1 z_2} &= \tfrac{1}{4}\big(E_{56} + E_{67}(+/-)(F_{56} + F_{67}) \\
&\qquad -i\big(E_{567} - E_6(+/-)(F_{567} - F_6)\big)\big)\,, \\
Z_{6\bar z_1 z_2} \,/\, \tfrac{1}{2}\mathcal{Z}_{6\bar z_1 z_2} &= \tfrac{1}{4}\big(E_{45^2678} + E_{4568}(+/-)(F_{45^2678} + F_{4568}) \\
&\qquad -i\big(E_{45^268} - E_{45678}(+/-)(F_{45^268} - F_{45678})\big)\big)\,,
\end{aligned}
\end{array}
\tag{6.4}
$$

and complex conjugates ($Q_A = -2$). Note that these five sectors are all different in $T^4/\mathbb{Z}_n$ for $n \geqslant 5$, while the two sectors with $Q_A = \pm 2$ will clearly have the same charge assignment in $T^4/\mathbb{Z}_4$. Finally, the orbifold $T^4/\mathbb{Z}_3$ merges, on the one hand, the two sectors with $Q_A = 2, -1$ and, on the other hand, the two remaining ones with $Q_A = 1, -2$, giving rise to three main sectors instead of five. In string theory, these three cases will lead to different twisted sectors, however, the untwisted sector and the residual U-duality algebra do not depend on n for any $n \geqslant 3$. The $n = 2$ case will again be treated separately.

For clarity, we will start by deriving the general structure of the (complex) invariant subalgebra, leaving aside, for the moment being, the analysis of its reality property. To do so, we perform a change of basis in the $Q_A = 0$ sector, separating raising from lowering operators. Let $X_\alpha = (1/2)(E_\alpha + F_\alpha)$ be the generator of any field element of $\mathfrak{g}^U$, we will resort to the combinations $X^+_\alpha \doteq X_\alpha + \frac{1}{2}\mathcal{X}_\alpha = E_\alpha$ and $X^-_\alpha \doteq X_\alpha - \frac{1}{2}\mathcal{X}_\alpha = F_\alpha$ to derive $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$. First, the following generators can be shown to form a basis of the non-abelian part of $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$:

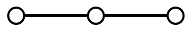
$$
\begin{aligned}
E_{\widetilde\alpha} &= -iE_{458} = Z^+_{6\bar z_2 z_2}\,, \qquad\qquad F_{\widetilde\alpha} = iF_{458} = Z^-_{6 z_2 \bar z_2}\,, \\
E_{\alpha_\pm} &= \frac{1}{2}\left(E_{56} - E_{67} \pm i(E_{567} + E_6)\right) = (K_{\bar z_1 \bar z_2}/K_{z_1 z_2})^+\,, \\
F_{\alpha_\pm} &= \frac{1}{2}\left(F_{56} - F_{67}\right) \pm i(-F_{567} - F_6)) = (K_{z_1 z_2}/K_{\bar z_1 \bar z_2})^-\,.
\end{aligned}
\tag{6.5}
$$

Computing their commutation relations determines the remaining generators of the algebra (for economy, we have omitted the lowering operators, which can be obtained quite straightforwardly by $F_\alpha = (E_\alpha)^\dagger$):

$$
\begin{aligned}
E_{\widetilde\alpha+\alpha_\pm} &\doteq \pm[E_{\widetilde\alpha}, E_{\alpha_\pm}] \;=\; \frac{1}{2}\left(E_{45^2678} - E_{4568} \pm i(E_{45678} + E_{45^268})\right) \\
&= (Z_{6\bar z_1 \bar z_2}/Z_{6 z_1 z_2})^+\,, \\
E_{\alpha_-+\widetilde\alpha+\alpha_+} &\doteq [E_{\alpha_-}, E_{\widetilde\alpha_0+\alpha_+}] \;=\; -iE_{45^26^278} = (Z_{6\bar z_1 z_1})^+\,, \\
H_{\widetilde\alpha} &\doteq [E_{\widetilde\alpha}, F_{\widetilde\alpha}] \;=\; (H_4 + H_5 + H_8)\,, \\
H_{\alpha_\pm} &\doteq [E_{\alpha_\pm}, F_{\alpha_\pm}] \;=\; \frac{1}{2}\big(H_5 + 2H_6 + H_7 \pm i(F_5 - E_5 + F_7 - E_7)\big)\,, \\
H_{\widetilde\alpha+\alpha_\pm} &\doteq [E_{\widetilde\alpha+\alpha_\pm}, F_{\widetilde\alpha+\alpha_\pm}] \\
&= \frac{1}{2}\big(2H_4 + 3H_5 + 2H_6 + H_7 + 2H_8 \mp i(E_5 - F_5 + E_7 - F_7)\big)\,, \\
H_{\alpha_-+\widetilde\alpha+\alpha_+} &\doteq [E_{\alpha_-+\widetilde\alpha+\alpha_+}, F_{\alpha_-+\widetilde\alpha+\alpha_+}] = H_4 + 2H_5 + 2H_6 + H_7 + H_8\,.
\end{aligned}
\tag{6.6}
$$

which shows that the non-abelian part of the complexified invariant subalgebra is of type $\mathfrak{a}_3 \simeq \mathfrak{d}_3$.

The rest of the $Q_A = 0$ sectors combines into two abelian contributions, so that the whole $D = 6$ $(\mathfrak{g}_{\rm inv})^{\mathbb{C}}$ reads

$\mathfrak{d}_3 \oplus \mathbb{C}^{\oplus^2}$:  $\times \ \{H_8 - H_4\} \ \times \ \{E_5 - F_5 - E_7 + F_7\}$

$\alpha_- \quad \widetilde{\alpha} \quad \alpha_+$

Concentrating on the non-abelian $\mathfrak{d}_3$ part of the real form $\mathfrak{g}_{\rm inv}$, we remark that it can be chosen to have Cartan subalgebra spanned by the basis $\left\{i(H_{\alpha_+} - H_{\alpha_-});\ H_{\widetilde{\alpha}};\ H_{\alpha_+} + H_{\alpha_-}\right\}$ compatible with the restriction $\mathrm{Fix}_{\tau_0}({\mathfrak{g}_{\rm inv}}^{\mathbb{C}})$. Since, in this basis, all ladder operators combine under ϕ into pairs of one compact and one non-compact operator, the signature of the real $\mathfrak{d}_3$ is again completely determined by the difference between non-compact and compact Cartan generators: since $i(H_{\alpha_+} - H_{\alpha_-})$ is compact while the two remaining generators are non-compact, $\sigma(\mathfrak{d}_3) = 1$, which determines the real form to be $\mathfrak{su}(2,2) \simeq \mathfrak{so}(4,2)$. The reality property of the invariant subalgebra is encoded in the Satake diagram of Table 9.

In addition, the two abelian factors appearing in the diagram above restrict, under Fix_{τ_0} to $H^{[4]} = H_8 - H_4$ and $i\widetilde{H}^{[4]} = (E_5 - F_5 - E_7 + F_7)$ and generate $\mathfrak{so}(1,1) \oplus \mathfrak{u}(1)$, similarly to the $T^2/\mathbb{Z}_{n>2}$ case. Their contributions to the signature cancel out, so $\sigma(\mathfrak{g}_{\rm inv}) = 1$

If we refer to the Satake diagram of Table 9, we note that in contrast to the split case, the arrows now joining the roots α_+ and α_- indeed change the compactness of the Cartan subalgebra without touching the "split" structure of the ladder operators. Moreover, the combinations $i(H_{\alpha_+} - H_{\alpha_-})$ and $H_{\alpha_+} + H_{\alpha_-}$ are now directly deducible from the action of ϕ on the set of simple roots.

Finally, as will be confirmed with the $T^6/\mathbb{Z}_{n>2}$ orbifold, if the chief inner automorphism $\mathcal{U}_q^{\mathbb{Z}_n}$ produces k pairs of Cartan generators in $(\mathfrak{g}_{\rm inv})^{\mathbb{C}}$ taking value in $\mathfrak{h}(\mathfrak{e}_{r|r}) \pm i\mathfrak{k}(\mathfrak{e}_{r|r})$, there will be k arrows joining the dual simple roots in the Satake diagram.

Compactifying further to $D = 5$, the additional node α_3 connects to $\widetilde{\alpha}$ forming a $\mathfrak{d}_4$ subalgebra. As in the $T^2/\mathbb{Z}_{n>2}$ case, this extra split $\mathfrak{a}_1$ will increase the total signature by one, yielding the real form $\mathfrak{so}(5,3)$. Since $\alpha_3(H_8 - H_4) \neq 0$, the non-compact Cartan generator $H^{[4]}$ commuting with $\mathfrak{so}(5,3) \oplus \mathfrak{u}(1)$ is now any multiple of $H^{[4]} = 2H_3 + 4H_4 + 3H_5 + 2H_6 + H_7$.

In $D = 4$, a new invariant root $\gamma = \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8 \in \Delta^+(\mathfrak{e}_7)$ appears which enhances the $\mathfrak{so}(1,1)$ factor to $sl(2,\mathbb{R})$. The reality property of the latter abelian factor can be checked by rewriting $\gamma = \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_- + \widetilde{\alpha} + \alpha_+$, which tells us that $\phi(\gamma) = -\gamma$. In $D = 3$, the additional node α_1 extending $\mathfrak{g}_{\rm inv}$ reconnects γ to the Dynkin diagram, resulting in $\mathfrak{so}(8,6) \oplus \mathfrak{u}(1)$.

6.1 Equivalence classes of involutive automorphisms of Lie algebras

Before treating the affine case, we shall introduce a procedure extensively used by [65], §14.4, to determine real forms by translating the adjoint action of the involutive automorphism on the generators by an exponential action on the root system directly.

Our concern in this paper will be only with real forms generated from chief *inner* involutive automorphisms, in other words involutions which can be written as $\vartheta = \mathrm{Ad}(e^{\overline{H}})$

for $\overline{H} \in \mathfrak{h}^{\mathbb{C}}$. In this case, a matrix realisation of the defining chief inner automorphism, denoted $\boldsymbol{\vartheta} = \mathrm{Ad}(e^{\overline{\mathbf{H}}})$ will act on the compact real form $\mathfrak{g}_c$ as:

$$\boldsymbol{\vartheta} = \mathbb{1}_r \oplus \sum_{\alpha\in\Delta_+} \begin{pmatrix} \cosh(\alpha(\overline{H})) & i\sinh(\alpha(\overline{H})) \\ -i\sinh(\alpha(\overline{H}) & \cosh(\alpha(\overline{H})) \end{pmatrix} , \qquad \forall\alpha \in \Delta_+ \, . \tag{6.7}$$

Being involutive $\boldsymbol{\vartheta}^2 = \mathbb{1}$ implies $\cosh(2\alpha(\overline{H})) = 1$, leading to

$$e^{\alpha(\overline{H})} = \pm 1, \quad \forall\alpha \in \Delta_+ \, . \tag{6.8}$$

In particular, this should hold for the simple roots: $e^{\alpha_i(\overline{H})} = \pm 1$, $\forall \alpha_i \in \Pi$. Then, how one assigns the $\pm$-signs to the simple roots completely determines the action of ϑ on the whole root lattice (6.8). For a r-rank algebra, there are then 2^r inner involutive automorphisms, but in general far less non-isomorphic real forms of $\mathfrak{g}$.

We are now ready to implement the procedure (4.9), first by splitting the positive root system Δ_+ into two subsets

$$\Delta_{(\pm 1)} \doteq \left\{\alpha \in \Delta_+ \mid e^{\alpha(\overline{H})} = \pm 1\right\} \tag{6.9}$$

and then by acting with the linear operator (4.8) in its matrix realisation (6.7) on the base of $\mathfrak{g}_c$. Then, the eigenspaces with eigenvalue (± 1) can be shown to be spanned by

$$\mathfrak{k} = \mathrm{Span}\left\{iH_{\alpha_j},\, \forall\alpha_j \in \Pi;\, (E_\alpha - F_\alpha) \text{ and } i(E_\alpha + F_\alpha),\ \forall\alpha \in \Delta_{(+1)}\right\} \tag{6.10}$$

and

$$\mathfrak{p} = \mathrm{Span}\left\{i(E_\alpha - F_\alpha),\, (E_\alpha + F_\alpha),\ \forall\alpha \in \Delta_{(-1)}\right\} \, . \tag{6.11}$$

In this approach, the signature determining all equivalence classes of involutive automorphisms (6.8) takes the handy form

$$-\sigma = \mathrm{Tr}\boldsymbol{\vartheta} = \big(r + 2\sum_{\alpha\in\Delta^+} \cosh(\alpha(\overline{H}))\big) = r + 2(\dim\Delta_{(+1)} - \dim\Delta_{(-1)}) \, . \tag{6.12}$$

6.2 A matrix formulation of involutive automorphisms of affine KMAs

This analysis can be extended to real forms of affine extension of Lie algebras. The general method based on a matrix reformulation of the involutive automorphism has been developed in [68] and successfully applied to the $\widehat{A}_r$, $\widehat{B}_r$, $\widehat{C}_r$ and $\widehat{D}_r$ cases in [69–73]. Here, we will only present the very basics of the method, and refer the reader to these articles for more details.

There are two ways of handling involutive automorphisms of untwisted affine Lie algebras. The first (classical) one is based on the study of Cartan-preserving automorphisms. Since every conjugacy class of the automorphism group contains at least one such automorphism, one can by this means arrive at a first classification of the involutive automorphisms of a given affine KMA. This procedure would be enough for determining all real forms of a finite Lie algebra, but would usually overcount them for affine KMA, because in this case some Cartan-preserving automorphisms can be conjugate via non-Cartan-preserving ones within $\mathrm{Aut}(\widehat{g})$. This will obviously reduce the number of conjugacy classes and by the same token the number of real forms of an untwisted affine KMA. A matrix formulation of automorphisms has been proposed in [68] precisely to treat these cases.

The first method takes advantage of the fact that Cartan-preserving automorphisms can be translated into automorphisms of the root system that leave the root structure of $\widehat{\mathfrak{g}}$ invariant. Let us call ϕ such an automorphism acting on $\Delta(\widehat{\mathfrak{g}})$. It can be constructed from an automorphism ϕ_0 acting on the basis of simple roots $\Pi(\mathfrak{g})$, for $\mathrm{rk}\mathfrak{g} = r$, as $\phi_0(\alpha_i) = \sum_{j=1}^{r}(\phi_0)_i^{\ j}\alpha_j$ for $i = 1, \ldots, r$. [12] Define the linear functional $\Omega \in P(\widehat{\mathfrak{g}})$ such that

$$\phi(\delta) = \mu\delta\,, \qquad \phi(\alpha_i) = \phi_0(\alpha_i) - (\phi_0(\alpha_i)|\Omega)\mu\delta\,,\ \forall i = 1, .., r\,. \tag{6.13}$$

with $(\alpha_i|\Omega) = n_i \in \mathbb{Z}$. This automorphism will be root-preserving if

$$\mu = \pm 1\,, \quad \text{and} \quad (\phi_0)_i^{\ j} \in \mathbb{Z}\,.$$

All root-preserving automorphisms can thus be characterized by the triple $\mathcal{D}_\phi = \{\phi_0, \Omega, \mu\}$, with the composition law:

$$\mathcal{D}_{\phi_1}\mathcal{D}_{\phi_2} = \{(\phi_1)_0 \cdot (\phi_2)_0, \mu_2\Omega_1 + (\phi_1)_0(\Omega_2), \mu_1\mu_2\} \tag{6.14}$$

The action of ϕ lifts to an algebra automorphism ϑ_ϕ. The first relation in expression (6.13) implies $\vartheta_\phi(c) = \mu c$, while we have:

$$\begin{aligned}
&\vartheta_\phi(z^n \otimes E_\alpha) = C_{\alpha+n\delta}\, z^{\mu(n-(\phi_0(\alpha_i)|\Omega))} \otimes E_{\phi_0(\alpha)}\,,\\
&\vartheta_\phi(z^n \otimes H_\alpha) = C_{n\delta}\, z^{\mu n} \otimes H_{\phi_0(\alpha)}\,, \qquad \vartheta_\phi(d) = \mu d + H_\Omega - \tfrac{1}{2}|\Omega|^2\mu c\,.
\end{aligned} \tag{6.15}$$

on the rest of the algebra. By demanding that ϑ_ϕ preserves the affine algebra (3.12), we can derive the relations, for α and $\beta \in \Delta(\mathfrak{g})$: $C_{n\delta}C_{m\delta} = C_{(m+n)\delta}$, $C_{\alpha+n\delta} = C_{n\delta}C_\alpha$, and $\mathcal{N}_{\alpha,\beta}C_{\alpha+\beta} = \mathcal{N}_{\phi_0(\alpha),\phi_0(\beta)}C_\alpha C_\beta$ with $C_0 = 1$ and $C_{-\alpha} = C_\alpha^{-1}$. The condition for ϑ_ϕ to be involutive is analogous to the requirement (6.8), namely

$$e^{\alpha_i(\overline{H})} = \pm 1\,, \qquad \forall i = 0, 1, .., r\,,$$

where $i = 0$ is this time included, and $\overline{H} = \sum c_i H_i + c_d d$, with $c_i, c_d \in \mathbb{C}$.

In particular, for a Cartan-preserving chief inner automorphism of type $\vartheta = e^{\mathrm{ad}(\overline{H})}$, we have:

$$\phi_0 = \mathbb{1}\,, \qquad \Omega = 0\,, \qquad \mu = 1\,, \qquad C_\alpha = e^{\alpha(\overline{H})}\,,\ \forall\alpha \in \Delta(\mathfrak{g})\,. \tag{6.16}$$

Possible real forms of an untwisted affine KMA are then determined by studying conjugacy classes of triples $\mathcal{D}_\phi$, for various involutive automorphisms ϕ. However, from the general structure (6.16), we see that a chief inner automorphism cannot be conjugate through a Cartan-preserving automorphism to an automorphism associated with a Weyl reflection, for instance. They could, however, be conjugate under some more general automorphism (note that this could not happen in the finite context). The above method might thus lead

[12] Not to confuse with the Cartan involution acting on the root system, as given from the Satake diagram. In the finite case, if ϕ_0 is non-trivial, it typically corresponds to outer automorphisms of the algebra.

to overcounting the number of equivalence classes of automorphisms, and consequently, of real forms of an affine Lie algebra.

This problem has been solved by a newer approach due to Cornwell, which is based on a matrix reformulation of the set of automorphisms for a given affine KMA. Choosing a faithfull d_Γ-dimensional representation of $\mathfrak{g}$ denoted by Γ, we can represent any element of $\mathcal{L}(\mathfrak{g})$ by $A(z) = \sum_{b=1}^{r}\sum_{n=-\infty}^{\infty} {a_n}^b\, z^n \otimes \Gamma(X_b)$, for $X_b \in \mathfrak{g}$. Then any element of $\widehat{\mathfrak{g}}$ may be written as:

$$\widehat{A}(z) = A(z) + \mu_c c + \mu_d d$$

where the $+$ are clearly not to be taken as matrix additions.

It has been pointed out in [68] that all automorphisms of complex untwisted KMAs are classified in this matrix formulation according to four types, christened: type 1a, type 1b, type 2a and type 2b.

A type 1a automorphism will act on $A(z)$ through an invertible $d_\Gamma \times d_\Gamma$ matrix $U(z)$ with components given by Laurent polynomials in z:

$$\varphi(A(z)) = U(z)A(uz)U(z)^{-1} + \frac{1}{\gamma_\Gamma}\oint \frac{dz}{2\phi i z}\mathrm{Tr}\left[\left(\frac{d}{dz}\ln U(z)\right)A(uz)\right]c\,, \tag{6.17}$$

where γ_Γ is the Dynkin index of the representation, and $u \in \mathbb{C}^*$ (this parameter corresponds, in the preceding formulation, to a Cartan preserving automorphism of type $\vartheta = e^{\mathrm{ad}(d)}$). The remaining three automorphisms are defined as above, by replacing $A(uz) \to \{-\widetilde{A}(uz); A(uz^{-1}); -\widetilde{A}(uz^{-1})\}$ on the RHS of expression (6.17) for, respectively, type {1b;2a;2b} automorphisms.

Here the tilde denotes the contragredient representation $-\widetilde{\Gamma}$. The action on c and d is the same for all four automorphisms, namely:

$$\begin{aligned}\varphi(c) &= \mu c\,,\\ \varphi(d) &= \mu\Phi(U(z)) + \lambda c + \mu d\,,\end{aligned}$$

with $\mu = 1$ for type 1a and 1b, and $\mu = -1$ for type 2a and 2b, and the matrix:

$$\Phi(U(z)) = -z\frac{d}{dz}\ln U(z) + \frac{1}{d_\Gamma}\mathrm{Tr}\left(z\frac{d}{dz}\ln U(z)\right)\mathbb{1}\,. \tag{6.18}$$

An automorphism φ can then be encoded in the triple: $\mathcal{D}_\varphi = \{U(z), u, \lambda\}$, and, as before, conjugation classes of automorphisms can be determined by studying equivalence classes of triples $\mathcal{D}_\varphi$. In this case, the more general structure of the matrix $U(z)$ as compared to ϕ_0, which acts directly on the generators of $\widehat{\mathfrak{g}}$ in a given representation, allows conjugation of two Cartan-preserving automorphisms via both Cartan-preserving and non-Cartan-preserving ones.

Finally, the conditions for φ to be involutive are, for type 1a:

$$u^2 = 1\,, \tag{6.19}$$

and

$$\begin{aligned}U(z)U(uz) &= \zeta z^k\mathbb{1}\,, \qquad \text{with } k \in \mathbb{N} \text{ and } \zeta \in \mathbb{C}\,,\\ \lambda &= -\frac{1}{2\gamma_\Gamma}\oint\frac{dz}{2\phi i z}\mathrm{Tr}\left[\left(\frac{d}{dz}\ln U(z)\right)\Phi(U(uz))\right]\,.\end{aligned} \tag{6.20}$$

For a type 1b automorphism, the first condition (6.19) remains the same, while we have to replace $U(uz) \to \widetilde{U}(uz)^{-1}$ and $\Phi(U(uz)) \to -\widetilde{\Phi}(U(uz))$ in the two last conditions (6.20).

Involutive automorphisms of type 2a and 2b are qualitatively different since they are already involutive for any value of u (so that condition (6.19) can be dropped), provided the last two conditions (6.20) are met, with the substitutions $U(uz) \to U(uz^{-1})$ in the first and $\Phi(U(uz)) \to \Phi(U(uz^{-1}))$ in the second one for type 2a, and $U(uz) \to \widetilde{U}(uz^{-1})^{-1}$ in the first and $\Phi(U(uz)) \to -\Phi(\widetilde{U}(uz^{-1}))$ in the second one for type 2b. In both cases, we are free to set $u = 1$.

When studying one particular class of involutive automorphisms, one will usually combine both the method based on root-preserving automorphisms and the one using the more elaborate matrix formulation to get a clearer picture of the resulting real form.

6.3 The non-split real invariant subalgebra in $D = 2$

The affine extension in $D = 2$ yields a real form of $\hat{\mathfrak{d}}_7 \bowtie \hat{\mathfrak{u}}(1)$. We will show that this real form, obtained from projecting from $\mathfrak{e}_{9|10}$ all charged states, builds a $\widehat{\mathfrak{so}}(8,6) \bowtie \hat{u}_{|1}(1)$, where, by $\widehat{\mathfrak{so}}(8,6)$, we mean the affine real form described by the $D = 2$ Satake diagram of Table 9 as determined in [74]. The proof requires working in a basis of $\mathfrak{g}_{\text{inv}}$ in which the Cartan subalgebra is chosen compact. It will be shown that such a basis can indeed be constructed from the restriction $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}} \cap \mathfrak{e}_{9|10}$. Then, by determining the action of ϕ on the latter, we will establish that, following [68], the vertex operator (*or* Sugawara) construction of $\mathfrak{g}_{\text{inv}}$ reproduces exactly the Cartan decomposition of $\widehat{\mathfrak{so}}(8,6)$ expected from [72]. Finally, we will show how the reality properties of $\hat{\mathfrak{d}}_7$, entail, through the affine central product, those of the $\hat{\mathfrak{u}}(1)$ factor.

Concentrating first on $\hat{\mathfrak{d}}_7$, we follow for a start the matrix method outlined in the preceding Section 6.2. In this case, the automorphism (6.17) restricted to the transformation $A(z) \to U(z)A(z)U(z)^{-1}$ has to preserve the defining condition:

$$A(z)^{\intercal}G + GA(z) = 0\,,$$

where G is the metric kept invariant by $SO(14)$ matrices in the rep Γ.

We start by choosing, for $\mathfrak{d}_7 \subset \hat{\mathfrak{d}}_7$, the 14-dimensional representation given in Appendix C with Dynkin index $\gamma_\Gamma = 1/\sqrt{42}$, whose generators will be denoted $\Gamma(E_\alpha)$ and $\Gamma(H_i)$. The affine extension of these operators is obtained as usual by the Sugawara construction, and the involutive automorphism $U(z)$ will be represented by a 14×14 matrix. This representation Γ is in fact equivalent to its contragredient one $-\widetilde{\Gamma}$ in the sense that one can find a 14×14 non-singular matrix C such that:

$$\Gamma(X) = -C\,\widetilde{\Gamma}(X)\,C^{-1}\,, \quad \forall X \in \mathfrak{d}_7\,.$$

One readily sees from eqn. (6.17) and subsequent arguments that, in this case, type 1b and 2b automorphisms coincide respectively with type 1a and 2a, which leaves us, for $\hat{\mathfrak{d}}_7$, with just two classes of involutive automorphisms, characterizing, roughly, real forms where the central charge c and the scaling operator d are both compact or both non-compact.

Since we do not expect the restriction ϕ of the Chevalley involution to $\mathfrak{g}_{\text{inv}}$ to mix levels in δ in this case, this in principle rules out all involutive automorphisms of type 2a, which explicitly depend on z. In turn, it tells us that the central charge and the scaling operator are now both compact in $\mathfrak{g}_{\rm inv}$, contrary to, for instance, the $T^7 \times T^2/\mathbb{Z}_{n>2}$ case analyzed in Section 5, and will be written $ic' \doteq iH_{\delta'_{D_7}} = ic$ and id'. Neither is the involution ϕ likely to involve different compactness properties for even and odd levels in δ. These considerations lead us to select $u = +1$. The z-independent automorphism of type 1a with $u = +1$ which seems to be a good candidate, in the sense that it reduces to $\mathfrak{so}(8,6)$ when we restrict to the finite Lie algebra $\mathfrak{d}_7 \subset \hat{\mathfrak{d}}_7$, is

$$U(z) = \mathbb{1}_4 \oplus (-\mathbb{1}_6) \oplus \mathbb{1}_4 \,, \tag{6.21}$$

so that eq.(6.17) reduces to $\varphi(A(z)) = U(z)A(z)U(z)^{-1}$.

Obviously, we have $\Phi(U(z)) = 0$ from expression (6.18) and the condition (6.20) for the automorphism to be involutive determines $\lambda = 0$. Now, since both central charge and scaling operator are compact in the new primed basis, we have $\mu = 1$. All these considerations put together lead to:

$$\varphi(ic') = ic' \,, \qquad \varphi(id') = id' \,, \tag{6.22}$$

from which we can determine the two triples:

$$\mathcal{D}_\varphi = \{\mathbb{1}_4 \oplus (-\mathbb{1}_6) \oplus \mathbb{1}_4; +1; 0\} \leftrightarrow \mathcal{D}_\phi = \{\mathbb{1}_7; 0; +1\} \,,$$

the structure of $\mathcal{D}_\phi$ clearly showing that we are dealing with a chief inner involutive automorphism. A natural choice for the primed basis of the Cartan subalgebra of $\mathfrak{d}_7$ is to pick it compact, so that its affine extension $\tilde{\mathfrak{h}} = \{iH'_1, \ldots, iH'_7, ic', id'\}$ is compact, as well.

We will check that the real form of $\hat{\mathfrak{d}}_7$ generated by the automorphism (6.21) and the one determined by the Cartan involution ϕ are conjugate, and thus lead to isomorphic real forms. Let us, for a start, redefine the basis of simple roots of $\hat{\mathfrak{d}}_7 \subset \mathfrak{g}_{\rm inv}$ appearing in Table 9:

$$\beta_1 \equiv \alpha_- \,, \quad \beta_2 \equiv \tilde{\alpha} \,, \quad \beta_3 \equiv \alpha_+ \,, \quad \beta_4 \equiv \alpha_3 \,, \quad \beta_5 \equiv \alpha_2 \,, \quad \beta_6 \equiv \alpha_1 \,, \quad \beta_7 \equiv \gamma \quad \beta_0 \equiv \alpha_0 \,. \tag{6.23}$$

The lexicographical order we have chosen ensures that the convention for the structure constants is a natural extension of the $D = 6$ case. As for $\mathfrak{e}_9$, we introduce an abbreviated notation $E_{\beta_6+2\beta_5+2\beta_4+\beta_1+2\beta_2+\beta_3} \doteq E_{\underline{65^24^212^23}}$. Conventions and a method for computing relevant structure constants are given in Appendix C.

We can now construct the compact Cartan subalgebra $\tilde{\mathfrak{h}}$ by selecting combinations of elements of $\mathfrak{d}_7 \subset \hat{\mathfrak{d}}_7 \subset \mathfrak{g}_{\rm inv}$ which commute and are themselves combinations of compact generators of $\mathfrak{e}_{9|10}$. The Cartan generators in this new basis are listed below, both in terms of $\mathfrak{d}_7 \subset (\mathfrak{g}_{\rm inv})^{\mathbb{C}}$ and $\mathfrak{e}_8 \subset \mathfrak{g}^U$ generators:

$$\begin{aligned}
iH'_{\underline{1}} &= i(E_{\underline{2}} + F_{\underline{2}}) \equiv E_{458} - F_{458} \,, \\
iH'_{\underline{2}} &= \frac{i}{2}\big(\eta_1(H_{\underline{3}} - H_{\underline{1}}) - E_{\underline{2}} - F_{\underline{2}} - E_{\underline{123}} - F_{\underline{123}}\big) \\
&\equiv \frac{1}{2}\big(\eta_1(E_5 - F_5 + E_7 - F_7) - E_{458} + F_{458} - E_{45^26^278} + F_{45^26^278}\big) \,,
\end{aligned}$$

$$
\begin{aligned}
iH'_{\underline{3}} &= i(E_{\underline{123}} + F_{\underline{123}}) \equiv E_{45^26^278} - F_{45^26^278}\,,\\
iH'_{\underline{4}} &= \frac{1}{2}\big(\eta_2(E_{\underline{54^212^23}} - F_{\underline{54^212^23}}) - i\eta_1(H_{\underline{3}} - H_{\underline{1}}) - E_{\underline{5}} + F_{\underline{5}}\big)\\
&\equiv -\frac{1}{2}\big(\eta_2(E_{234^25^36^278^2} - F_{234^25^36^278^2}) + \eta_1(E_5 - F_5 + E_7 - F_7) + E_2 - F_2\big)\,,\\
iH'_{\underline{5}} &= E_{\underline{5}} - F_{\underline{5}} \equiv E_2 - F_2\,,\\
iH'_{\underline{6}} &= \frac{1}{2}\big(\eta_3(E_{\underline{76^25^24^212^23}} - F_{\underline{76^25^24^212^23}}) - E_{\underline{7}} + F_{\underline{7}} - E_{\underline{5}} + F_{\underline{5}} - \eta_2(E_{\underline{54^212^23}} - F_{\underline{54^212^23}})\big)\\
&\equiv -\frac{1}{2}\big(\eta_3(E_{\theta_{E_8}} - F_{\theta_{E_8}}) + E_\gamma - F_\gamma + E_2 - F_2 - \eta_2(E_{234^25^36^278^2} - F_{234^25^36^278^2})\big)\,,\\
iH'_{\underline{7}} &= E_{\underline{7}} - F_{\underline{7}} \equiv E_\gamma - F_\gamma\,,
\end{aligned}
\tag{6.24}
$$

where the factors $\eta_i = \pm 1$, $\forall i = 1, 2, 3$, determine equivalent solutions.

The Cartan generator attached to the affine root β'_0 is constructed from the above (6.24) in the usual way:

$$
\begin{aligned}
iH'_{\underline{0}} &= iH_{\delta'_{D_7}} - \eta_3(E_{\underline{76^25^24^212^23}} - F_{\underline{76^25^24^212^23}})\\
&= ic' + \eta_3(E_{\theta_{E_8}} - F_{\theta_{E_8}})\,,
\end{aligned}
$$

which commutes with $\tilde{\mathfrak{h}}$ (6.24) and is indeed compact, as expected from expression (6.22).

We find the associated ladder operators by solving the set of equations $[H'_{\underline{j}}, E'_{\underline{i}}] = A_{\underline{ij}} E'_{\underline{i}}$, $[E'_{\underline{i}}, F'_{\underline{j}}] = \delta_{\underline{ij}} H'_{\underline{i}}$ and $[E'_{\underline{i}}, E'_{\underline{j}}] = \mathcal{N}_{\underline{i},\underline{j}} E'_{\underline{ij}}$ (the corresponding commutation relations for the lowering operators are then automatically satisfied). Here we write $E'_{\underline{i}} \equiv E_{\beta'_i}$ and $F'_{\underline{i}} \equiv E_{-\beta'_i}$ for short, for the set $\Pi' = \{\beta'_0, \ldots, \beta'_7\}$ of simple roots dual to the Cartan basis (6.24). Thus:

$$
\begin{aligned}
E'_{\underline{1}}/F'_{\underline{1}} &= H_{\underline{2}} \mp (E_{\underline{2}} - F_{\underline{2}}) \equiv H_{458} \pm i(E_{458} + F_{458})\,,\\
E'_{\underline{2}} &= E_{\underline{1}} - F_{\underline{3}} - \eta_1(F_{\underline{12}} - F_{\underline{23}}) \equiv E_{\alpha_-} - F_{\alpha_+} - \eta_1(E_{\alpha_- + \tilde{\alpha}} - F_{\tilde{\alpha}+\alpha_+})\,,\\
E'_{\underline{3}}/F'_{\underline{3}} &= H_{\underline{123}} \mp (E_{\underline{123}} - F_{\underline{123}}) \equiv H_{45^26^278} \pm i(E_{45^26^278} + F_{45^26^278})\,,\\
E'_{\underline{4}} &= E_{\underline{4(23\leftrightarrow12)}} + iE_{\underline{54(23\leftrightarrow12)}} - \eta_2(F_{\underline{4(12\leftrightarrow23)}} + iF_{\underline{54(12\leftrightarrow23)}})\,, \quad \text{for } \eta_1 = \pm 1\,,\\
E'_{\underline{5}}/F'_{\underline{5}} &= H_{\underline{5}} \pm i(E_{\underline{5}} + F_{\underline{5}}) \equiv H_2 \pm i(E_2 + F_2)\,,
\end{aligned}
\tag{6.25}
$$

together with $F'_i = (E'_i)^\dagger$, for $i = 2, 4$. In the expression in (6.25) for $E'_{\underline{4}}$, the $\leftrightarrow$ gives the two possible values of the last two indices depending on the choice of $\eta_1 = \pm 1$. It can be checked that $[E'_{\underline{2}}, E'_{\underline{4}}] = 0$.

The raising operator E'_6 is independent of η_1 and takes the form:

$$
\begin{aligned}
E'_{\underline{6}} = {} & (E_{\underline{6}} + iE_{\underline{65}} - iE_{\underline{76}} + E_{\underline{765}}) + \eta_2\eta_3(F_{\underline{6}} + iF_{\underline{65}} - iF_{\underline{76}} + F_{\underline{765}})\\
& - \eta_2(iE_{\underline{654^212^23}} + E_{\underline{65^24^212^23}} + E_{\underline{7654^212^23}} - iE_{\underline{765^24^212^23}})\\
& - \eta_3(iF_{\underline{654^212^23}} + F_{\underline{65^24^212^23}} + F_{\underline{7654^212^23}} - iF_{\underline{765^24^212^23}})\,,
\end{aligned}
\tag{6.26}
$$

the corresponding lowering operator is obtained from the above by hermitian conjugation. Moreover, it can be verified after some tedious algebra that indeed $[E'_{\underline{4}}, E'_{\underline{6}}] = 0$. Note that we have translated the primed generators into $\mathfrak{e}_8$ ones only when the expression is not too lengthy. Hereafter, such substitutions will be made only when necessary.

The two remaining pairs of ladder operator enhancing $\mathfrak{d}_6$ to $\hat{\mathfrak{d}}_7$ are:

$$
\begin{aligned}
E'_{\underline{7}}/F'_{\underline{7}} &= H_{\underline{7}} \pm i(E_{\underline{7}} + F_{\underline{7}}) \equiv H_\gamma \pm i(E_\gamma + F_\gamma)\,, \\
E'_{\underline{0}}/F'_{\underline{0}} &= -\eta_3\, t^{\pm 1} \otimes \big(H_{\underline{\theta}_{D_7}} \mp i(E_{\underline{\theta}_{D_7}} + F_{\underline{\theta}_{D_7}})\big) \\
&\equiv \eta_3\, t^{\pm 1} \otimes \big(H_{\theta_{E_8}} \mp i(E_{\theta_{E_8}} + F_{\theta_{E_8}})\big)\,,
\end{aligned}
\tag{6.27}
$$

where $\underline{\theta}_{D_7} \doteq \beta_7 + 2(\beta_6 + \beta_5 + \beta_4 + \beta_2) + \beta_1 + \beta_3$.

At this stage it is worth pointing out that the affine real form $\mathfrak{g}_{\text{inv}}$ is realized as usual as a central extension of the loop algebra of the finite $(\mathfrak{d}_7)_0$ which may or may not descend to a real form of $\mathfrak{d}_7$ (in our case, it does since it will be shown that $(\mathfrak{d}_7)_0 = \mathfrak{d}_{7|5}$)

$$
\mathfrak{g}_{\text{inv}}/\mathcal{L}(\mathfrak{u}(1)) = \mathbb{R}[t, t^{-1}] \otimes (\mathfrak{d}_7)_0 \oplus \mathbb{R}ic' \oplus \mathbb{R}id'\,.
$$

The difference is that we are now tensoring with an algebra of Laurent polynomials $\mathcal{L} = \mathbb{R}[t, t^{-1}]$ in the (indeterminate) variable t defined as follows

$$
t = \frac{1}{2}\big((1-i) + (1+i)\vartheta_C\big) z \equiv \frac{1}{1+i}\big(1 + i\vartheta_C\big) z\,. \tag{6.28}
$$

The second term of the equality (6.28) is clearly reminescent from the operator $\sqrt{\vartheta}$ (4.8). The inverse transformation yields:

$$
z = \frac{(1+i)t + \sqrt{2i(t^2-2)}}{2}\,, \qquad z^{-1} \equiv \bar{z} = \frac{(1+i)t - \sqrt{2i(t^2-2)}}{2i}\,.
$$

On can check that under the Chevalley involution: $\vartheta_C(t) = t$ and $\vartheta_C(t^{-1}) = t^{-1}$. Moreover, using

$$
t^n = \frac{1}{(1-i)^n z^n} \sum_{k=0}^{n} \binom{n}{k} (-iz^2)^k
$$

one can check that $\vartheta_C(t^n) = t^n\ \forall n \in \mathbb{Z}^*$, as required by the affine extension of the basis (6.24-6.27), which will become clearer when we give the complete realization of the real $\mathfrak{g}_{\text{inv}}/\mathcal{L}(u(1))$ (6.33-6.34).

Finally, we may now give the expression of the compact scaling operator in the primed basis:

$$
id' = \frac{(1+i\vartheta_C)z}{(1-i\vartheta_C)z}\, id\,,
$$

which can be shown to be Hermitian.

Now that we know the structure of the generators $E'_{\underline{i}}$ and $F'_{\underline{i}}$, $i = 0, \ldots, 7$, we are in the position of determining $\text{Fix}_{\tau_0}(\hat{\mathfrak{d}}_7)$ and, by acting with ϕ on the latter, are able to reconstruct the eigenvalues of the representation $\phi = \text{Ad}(e^{\overline{H}})$ of the Cartan involution on the basis (6.10)-(6.11), namely $\phi \cdot (\hat{\mathfrak{d}}_7)_{\beta'} = e^{\beta'(\overline{H})}(\hat{\mathfrak{d}}_7)_{\beta'}$, with $e^{\beta'_i(\overline{H})} = \pm 1$, $\forall \beta'_i \in \Pi'$. We will then show that the four automorphisms determined by this method corresponding to all possible values of η_i, $i = 1, 2, 3$, are conjugate to the action of the $U(z)$ given in (6.21) on the representation Γ of $\mathfrak{d}_7$.

Reexpressing, for instance, the second line of the list (6.25) in terms of the original basis (6.5) and (6.6), and taking $\mathrm{Fix}_{\tau_0}(\hat{\mathfrak{d}}_7)$ yields the two following generators of $\mathfrak{g}_{\mathrm{inv}}$:

$$\begin{aligned}
\frac{1}{2}(E'_{\underline{2}} - F'_{\underline{2}}) &= \frac{1}{2}\Big(E_{56} - F_{56} - E_{67} + F_{67} - \eta_1\big(E_{45^26^278} - F_{45^26^278} - E_{4568} + F_{4568}\big)\Big)\,,\\
\frac{i}{2}(E'_{\underline{2}} + F'_{\underline{2}}) &= \frac{1}{2}\Big(E_{567} - F_{567} + E_6 - F_6 - \eta_1\big(E_{45678} - F_{45678} + E_{45^2678} - F_{45^268}\big)\Big)\,.
\end{aligned} \tag{6.29}$$

Both are obviously invariant under ϕ, since they are linear combinations of compact generators. According to Section 6.8, we have $e^{\beta'_2(\overline{H})} = +1$. The same reasoning applies to the pairs $E'_{\underline{4}}/F'_{\underline{4}}$ and $E'_{\underline{6}}/F'_{\underline{6}}$. In contrast to the $E'_{\underline{2}}/F'_{\underline{2}}$ case, these two couples of generators will be alternatively compact or non-compact depending on the sign of η_2 and η_3. In particular, since $E'_{\underline{4}}$ has basic structure $[E_\alpha, E_{\alpha\pm}] - \eta_2[F_{\alpha\mp}, F_\alpha]$, the choice $\eta_2 = +1$ will produce the two compact combinations $2^{-1}(E'_{\underline{4}} - F'_{\underline{4}})$ and $2^{-1}i(E'_{\underline{4}} + F'_{\underline{4}})$, while the opposite choice selects the two non-compact ones, by flipping the reciprocal sign between E and F. From expression (6.26), we see that the $E'_{\underline{6}}/F'_{\underline{6}}$ case is even more straightforward, compactness and non-compactness being selected by $\eta_2\eta_3 = \pm 1$ respectively. At this stage, our analysis thus leads to the four possibilities: $e^{\beta'_4(\overline{H})} = \pm 1$ and $e^{\beta'_6(\overline{H})} = \pm 1$.

Finally, the remaining ladder operators $E'_{\underline{i}}$ and $F'_{\underline{i}}$ for $i = 0, 1, 3, 5, 7$ combine in purely non-compact expressions, for instance

$$\frac{1}{2}(E'_{\underline{1}} + F'_{\underline{1}}) = H_{458}\,, \qquad \frac{i}{2}(E'_{\underline{1}} - F'_{\underline{1}}) = -(E_{458} + F_{458})\,. \tag{6.30}$$

The $E'_{\underline{0}}/F'_{\underline{0}}$ case is a bit more subtle because of the presence of the (t, t^{-1}) loop factors, and requires adding $E_{\theta_{D_7}+\delta}/F_{\theta_{D_7}+\delta} = \eta_3\, t^{\pm 1} \otimes \big(H_{\theta_{E_8}} \pm i(E_{\theta_{E_8}} + F_{\theta_{E_8}})\big)$ into the game. Computing Fix_{τ_0} for all of these four operators results in four non-compact combinations. This is in accordance with θ'_{D_7} which we now know to statisfy $e^{\theta'_{D_7}(\overline{H})} = -1$ for all four involutive automorphisms, and tells us in addition that: $e^{\beta'_0(\overline{H})} = -1$.

Collecting all previous results, the eigenvalues of the four involutive automorphisms $\phi_{(\eta_2,\eta_3)} = \mathrm{Ad}(e^{\overline{H}_{(\eta_2,\eta_3)}})$ are summarized in the table (6.31) below.

η_2	η_3	$e^{\beta'_2(\overline{H})}$	$e^{\beta'_4(\overline{H})}$	$e^{\beta'_6(\overline{H})}$	$e^{\beta'_{i\neq 2,4,6}(\overline{H})}$
$+1$	-1	$+1$	$+1$	$+1$	-1
$+1$	$+1$	$+1$	$+1$	-1	-1
-1	$+1$	$+1$	-1	$+1$	-1
-1	-1	$+1$	-1	-1	-1

(6.31)

The Cartan element $\overline{H}$ defining the involution ϕ can be read off table (6.31). The most general solution is given by $\overline{H} = \pi i \sum_{i=0}^{7} c_i H'_{\underline{i}} + \pi i c_{d'} d'$ with

$$\begin{gathered}
c_1 = c_3 = \frac{\kappa+1}{2} + \mathbb{Z}\,, \quad c_4 = \kappa - 1\,, \quad c_5 = \kappa - \frac{\eta_2 - 1}{2} + \mathbb{Z}\,, \quad c_6 = \kappa + \eta_2 - 1\,,\\
c_7 = \frac{\kappa + \eta_2}{2} + \mathbb{Z}\,, \quad c_0 = \frac{\kappa + \eta_2 - \eta_3 + 1}{2} + \mathbb{Z}\,, \quad c_d = \eta_3 - 1
\end{gathered}$$

where $c_2 \doteq \kappa \in \mathbb{C}$ is a free parameter.

Restricted to $\mathfrak{d}_7$, the four inner automorphisms defined in the table (6.31) are all in the same class of equivalence, and thus determine the same real form, namely $\mathfrak{so}(8,6)$ as expected from $\mathfrak{g}_{\rm inv}$ in $D=3$. In Appendix C, we have computed the two sets of roots $\Delta_{(+1)}$ and $\Delta_{(-1)}$ (6.9) generating the Cartan decomposition (6.10)–(6.11) of the real form. It can be checked that, in these four cases, the signature $\sigma|_{\mathfrak{d}_7} = -(7+2(\dim\Delta_{(+1)}-\dim\Delta_{(+1)})) = 5$, in accordance with $\mathfrak{so}(8,6)$.

The involutive automorphism (6.21) in turn can be shown to split the root system of $\mathfrak{d}_7$ according to

$$\beta' = \varepsilon_i \pm \varepsilon_j \rightarrow \begin{cases} \beta' \in \Delta_{(+1)} \;\; 1 \leqslant i < j \leqslant 4 \text{ and } 4 < i < j \leqslant 7 \\ \beta' \in \Delta_{(-1)} \qquad\quad 1 \leqslant i \leqslant 4 < j \leqslant 7 \end{cases}, \tag{6.32}$$

which can be verified by computing $U(z)\gamma(E_\alpha)U(z)^{-1}$ for the representation γ (C.4). We can check that we have again: $\sigma|_{\mathfrak{d}_7} = 5$, for the splitting (6.32), since the automorphism (6.21) corresponds, in our previous formalism to the involution $e^{\beta'_4(\overline{H})} = -1$ and $e^{\beta'_i(\overline{H})} = +1$, $\forall i \neq 4$.

Since they are conjugate at the $\mathfrak{d}_7$ level and all of them preserve the central charge and scaling element, the four automorphisms (6.31) lift to conjugate automorphisms of $\hat{\mathfrak{d}}_7$. All four of them are again clearly conjugate to $U(z)$ defined by properties (6.21), (6.22) and (6.32). These five Cartan preserving inner involutive automorphisms lead to equivalent Cartan decompositions $\mathfrak{k} \oplus^\perp \mathfrak{p}$ (4.6) given by generalizing the basis (6.29) and (6.30) found previously to the affine case:

$$\begin{aligned}
\mathfrak{k} : \bullet\, & iH'_{\underline{k}} \; (\forall k = 1,..,7);\; ic';\; id'; \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (6.33)\\
\bullet\, & \frac{1}{2}(t^n - t^{-n}) \otimes H'_{\underline{k}} \quad \text{and} \quad \frac{i}{2}(t^n + t^{-n}) \otimes H'_{\underline{k}} \qquad (\forall k = 1,..,7;\; n \in \mathbb{N}^*)\,;\\
\bullet\, & \frac{1}{2}(t^n \otimes E_{\beta'} - t^{-n} \otimes F_{\beta'}) \quad \text{and} \quad \frac{i}{2}(t^n \otimes E_{\beta'} + t^{-n} \otimes F_{\beta'})\,,\; n \in \mathbb{Z}\\
& \qquad (\forall \beta' \in \Delta_{(+1)} \text{ defined by (6.32), } (C.5), (C.6), (C.7) \text{ and } (C.8))\\
\mathfrak{p} : \bullet\, & \frac{i}{2}(t^n \otimes E_{\beta'} - t^{-n} \otimes F_{\beta'}) \quad \text{and} \quad \frac{1}{2}(t^n \otimes E_{\beta'} + t^{-n} \otimes F_{\beta'}),\; n \in \mathbb{Z} \qquad\qquad (6.34)\\
& \qquad (\beta' \in \Delta_{(-1)} = \Delta_+(D_7)\backslash\Delta_{(+1)})\,.
\end{aligned}$$

These decompositions define isomorphic real forms, which we denote by $\widehat{\mathfrak{so}}(8,6)$, encoded in the affine Satake diagram of Table 9 (see for instance [74] for a classification of untwisted and twisted affine real forms).

We have checked before the behaviour of the ladder operators of the finite $\mathfrak{d}_7$ subalgebra of $\mathfrak{g}_{\rm inv}$. The verification can be performed in a similar manner for the level $n \geqslant 1$ roots $\beta' + n\delta'_{D_7}$. Applying for example ${\rm Fix}_{\tau_0}$ to the four generators $2^{-1}(t^{\pm n} \otimes E_{\underline{4}} - t^{\mp n} \otimes F_{\underline{4}})$ and $2^{-1}i(t^{\pm n} \otimes\; E_{\underline{4}} +\; t^{\mp n} \otimes F_{\underline{4}})$, one obtains the following combinations

$$\begin{aligned}
&\tfrac{1}{2}(t^n + t^{-n}) \otimes \big(E_{345^2678} - \eta_2 F_{345^2678} - E_{34568} + \eta_2 F_{34568}\\
&\qquad\qquad\qquad\qquad -E_{2345678} + \eta_2 F_{2345678} - E_{2345^268} + \eta_2 F_{2345^268}\big)\,,\\
&\tfrac{1}{2}(t^n - t^{-n}) \otimes \big(E_{345678} - \eta_2 F_{345678} + E_{345^268} - \eta_2 F_{345^268}\\
&\qquad\qquad\qquad\qquad +E_{2345^2678} - \eta_2 F_{2345^2678} - E_{234568} + \eta_2 F_{234568}\big)\,,
\end{aligned}$$

which, since now $\vartheta_C(t^n \pm t^{-n}) = t^n \pm t^{-n}$, are all either non-compact if $\eta_2 = +1$, or compact otherwise, by virtue of table (6.31). This is in accordance with the Cartan decomposition (6.33-6.34). The compactness of the remaining $n \geqslant 1$ ladder operators can be checked in similar and straightforward fashion by referring once again to the table (6.31).

In contrast to the split case, a naive extension of the signature, which we denote by $\hat{\sigma}$, is not well defined since it yields in this case an infinite result:

$$\hat{\sigma} = 3 + 2 \times 5 \times \infty\,. \tag{6.35}$$

In the first finite contribution, we recognize the signature of $\mathfrak{so}(8,6)$ together with the central charge and scaling element, while the infinite towers of vertex operator contribute the second part. As mentioned before in the $D = 4$ case, the signature for the finite $\mathfrak{d}_7$ amounts to the difference between compact and non-compact Cartan generators, for the following alternative choice of basis for the Cartan algebra $\{H_\gamma; H_1; H_2; H_3; H_{\tilde{\alpha}}; H_{\alpha_+} + H_{\alpha_-}; i(H_{\alpha_+} - H_{\alpha_-})\}$. This carries over to the infinite contribution in expression (6.35), where it counts the number of overall compact towers, with an additional factor of 2 coming from the presence of both raising and lowering operators.

Care must be taken when defining the real affine central product $\mathfrak{g}_{\text{inv}}$. The real Heisenberg algebra

$$\hat{\mathfrak{u}}(1)_{|1} = \sum_{n=0}^{\infty} \mathbb{R}(z^n + z^{-n}) \otimes i\widetilde{H}^{[4]} + \sum_{n=1}^{\infty} \mathbb{R}(z^n - z^{-n}) \otimes \widetilde{H}^{[4]} + \mathbb{R}c + \mathbb{R}d$$

is in this case isomorphic to the one appearing in the $T^2/\mathbb{Z}_{n>2}$ orbifold (5.6). Clearly both scaling operators and central charge are, in contrast to $\widehat{\mathfrak{so}}(8,6)$ non-compact. The identification required by the affine central product formally takes place before changing basis in $\widehat{\mathfrak{so}}(8,6)$ to the *primed* operators. The central charge and scaling operators acting on both subspaces of $\mathfrak{g}_{\text{inv}} = \widehat{\mathfrak{so}}(8,6) \bowtie \hat{\mathfrak{u}}_{|1}(1)$ are then redefined as $d \oplus d \to id' \oplus d$ and $c \oplus c \to ic' \oplus c \equiv ic \oplus c$. Then we can write $\mathfrak{g}_{\text{inv}} = \widehat{\mathfrak{so}}(8,6) \oplus \hat{\mathfrak{u}}_{|1}(1)/\{\mathfrak{z}, \bar{d}\}$, with $\mathfrak{z} = c - c'$ and $\bar{d} = d - \frac{\sqrt{2i(t^2-2)}}{(1+i)t} d'$. The signature $\hat{\sigma}$ of $\mathfrak{g}_{\text{inv}}$ is undefined.

6.4 The non-split real Borcherds symmetry in $D = 1$

The analysis of the $D = 1$ invariant subalgebra closely resembles the $T^8 \times T^2/\mathbb{Z}_{n>2}$ case. The central product of Section 6.3 is extended to a direct sum of a $\mathfrak{u}(1)$ factor with the quotient of a Borcherds algebra by an equivalence relation similar to the one stated in Conjecture 5.1. The Borcherds algebra ${}^4\mathcal{B}_{10}$ found here is defined by a 10×10 degenerate Cartan matrix of rank $r = 9$. Its unique isotropic imaginary simple root (of multiplicity one) ξ_I is now attached to the raising operator $E_{\xi_I} = (1/2)(E_{\delta+\alpha_5} - E_{\delta-\alpha_5} - E_{\delta+\alpha_7} + E_{\delta-\alpha_7})$, so that the equivalence relation defining $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$ from ${}^4\mathcal{B}_{10} \oplus \mathfrak{u}(1)$ identifies the Cartan generator $H_I \doteq H_{\xi_I}$ with H_δ and removes the derivation operator $d_I \doteq d_{\xi_I}$.

Moreover, the splitting of multiplicities should occur as in the $T^8 \times T^2/\mathbb{Z}_{n>2}$ example, since $\dim(\hat{\mathfrak{d}}_7)_{\delta_{D_7}} = \dim(\hat{\mathfrak{e}}_7)_\delta$. It might a priori seem otherwise from the observation that both $\widetilde{K}_{(7)[2\cdots 679\,10]} - \widetilde{K}_{(8)[2\cdots 689\,10]}$ and $\widetilde{K}_{(9)[2\cdots 89]} - \widetilde{K}_{(10)[2\cdots 8\,10]}$ are separately invariant.

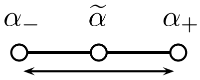
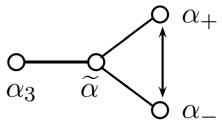
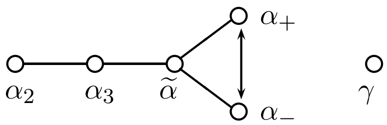
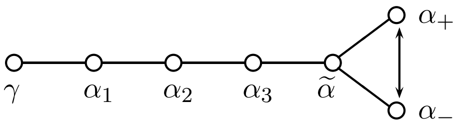
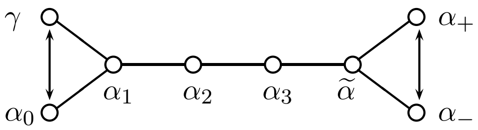
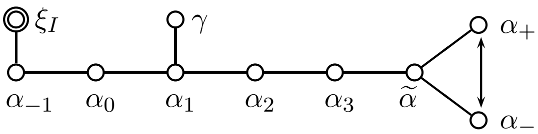
D	(Π_0, ϕ)	$\mathfrak{g}_{\text{inv}}$	$\sigma(\mathfrak{g}_{\text{inv}})$
6	 $\times \quad H^{[4]} \times \quad i\widetilde{H}^{[4]}$	$\mathfrak{so}(4,2) \oplus \mathfrak{so}(1,1) \oplus \mathfrak{u}(1)$	1
5	 $\times \quad H^{[4]} \times \quad i\widetilde{H}^{[4]}$	$\mathfrak{so}(5,3) \oplus \mathfrak{so}(1,1) \oplus \mathfrak{u}(1)$	2
4	 $\times \quad i\widetilde{H}^{[4]}$	$\mathfrak{so}(6,4) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{u}(1)$	3
3	 $\times \quad i\widetilde{H}^{[4]}$	$\mathfrak{so}(8,6) \oplus \mathfrak{u}(1)$	4
2	 $\times \quad \left\{ i\widetilde{H}^{[4]}_n \right\}_{n\in\mathbb{Z}}$	$\widehat{\mathfrak{so}}(8,6) \oplus \mathcal{L}(\mathfrak{u}(1))_{\vert -1}$	-
1	 $\times \quad i\widetilde{H}^{[4]}$	${}^4\mathcal{B}_{10(Ib)} \oplus \mathfrak{u}(1)$	-

Table 9: The real subalgebras $\mathfrak{g}_{\text{inv}}$ for $T^{7-D} \times T^4/\mathbb{Z}_{n>2}$ compactifications

However, the combination:

$$\widetilde{K}_{(7)[2\cdots 679\,10]} - \widetilde{K}_{(8)[2\cdots 689\,10]} + \widetilde{K}_{(9)[2\cdots 89]} - \widetilde{K}_{(10)[2\cdots 8\,10]} = \widetilde{K}_{[2\cdots 10]} \otimes i(H_{\alpha_+} - H_{\alpha_-}) \in \mathfrak{d}_7^{\wedge\wedge}$$

contributes to the multiplicity of δ, while we may rewrite

$$\frac{1}{2}(\widetilde{K}_{(7)[2\cdots 679\,10]} - \widetilde{K}_{(8)[2\cdots 689\,10]} - \widetilde{K}_{(9)[2\cdots 89]} + \widetilde{K}_{(10)[2\cdots 8\,10]}) = E_{\xi_I} \in {}^4\mathcal{B}_{10}\,,$$

which is the unique raising operator spanning $({}^4\mathcal{B}_{10})_{\xi_I}$. Thus, though root multiplicities remain unchanged, we have to group invariant objects in representations of $\mathfrak{sl}(6,\mathbb{R})$, which are naturally shorter than in the $T^2/\mathbb{Z}_n$ case. We will not detail all such representations here, since they can in principle be reconstructed by further decomposition and/or regrouping of the results of Table 7.

The real invariant subalgebra can again be formally realized as

$$\mathfrak{g}_{\rm inv} = \mathfrak{u}(1) \oplus {}^4\mathcal{B}_{10(Ib)}/\{\mathfrak{z}, d_I\}$$

where $\mathfrak{z} = H_\delta - H_I$. We denote by $\mathcal{B}_{10(Ib)}$ the real Borcherds algebra obtained from $\mathrm{Fix}_{\tau_0}({}^4\mathcal{B}_{10})$ and represented in Table 9, choosing Ia to refer to the split form. The disappearance of the diagram automorphism which, in the $D=2$ case, exchanged the affine root α_0 with γ leads to non-compact H_{-1} and H_δ, in contrast to what happened with the $\widehat{\mathfrak{so}}(8,6) \subset \mathfrak{g}_{\rm inv}$ factor in $D=2$. This is reflected by ξ_I being a white node with no arrow attached to it. Note that a black isotropic imaginary simple root connected to a white real simple root would, in any case, be forbidden, since such a diagram is not given by an involution on the root system. Moreover, an imaginary simple root can only be identified by an arrow to another imaginary simple root (and similarly for real simple roots).

7. The orbifolds $T^6/\mathbb{Z}_{n>2}$

The orbifold compactification $T^{5-D} \times T^6/\mathbb{Z}_n$ for $n \geqslant 3$ can be carried out similarly. We fix the orbifold action in the directions $\{x^5, x^6, x^7, x^8, x^9, x^{10}\}$, so that it will only be felt by the set of simple roots $\{\alpha_2, .., \alpha_8\}$ defining the $\mathfrak{e}_{7|7}$ subalgebra of $\mathfrak{g}^U = \mathrm{Split}(\mathfrak{e}_{11-D})$ from $D=4$ downward. Thus, we may start again by constructing the appropriate charged combinations of generators for $\mathfrak{g}^U = \mathfrak{e}_{7|7}$, and then extend the result for $D \leq 3$ in a straightforward fashion. Since $\mathfrak{e}_7$ has 63 positive roots, we will restrict ourselves to the invariant subalgebra, and list only a few noteworthy charged combinations of generators. In this case, a new feature appears: the invariant algebra is not independent of n, $\forall n \geqslant 3$, as before. Instead, the particular cases $T^6/\mathbb{Z}_3$ and $T^6/\mathbb{Z}_4$ are non-generic and yield invariant subalgebras larger than the $n \geqslant 5$ one.

More precisely, we start by fixing the orbifold action to be

$$(z_i, \bar{z}_i) \to (e^{2\pi i/n} z_i, e^{-2\pi i/n} \bar{z}_i) \quad \text{for } i = 1, 2, \qquad (z_3, \bar{z}_3) \to (e^{-4\pi i/n} z_3, e^{4\pi i/n} \bar{z}_3)\,, \tag{7.1}$$

in other words, we choose $Q_1 = +1$, $Q_2 = +1$ and $Q_3 = -2$ to ensure $\sum_i Q_i = 0$. Note that for values of n that are larger than four, there are other possible choices, like $Q_1 = 1$, $Q_2 = 2$ and $Q_3 = -3$ for $T^6/\mathbb{Z}_6$ or $Q_1 = 1$, $Q_2 = 3$ and $Q_3 = -4$ for $T^6/\mathbb{Z}_8$ and so on. Indeed, the richness of T^6/Z_n orbifolds compared to T^4/Z_n ones stems from these many possibilities. Though interesting in their own right, we only treat the first of the above cases in detail, though any choice of charges can in principle be worked out with our general method. One has to keep in mind, however, that any other choice than the one we made in expression (7.1) may lead to different non-generic values of n.

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7.1 The generic $n \geqslant 5$ case

Concentrating on the invariant subalgebra $(\mathfrak{g}_{\rm inv})^{\mathbb{C}}$ for $n \geqslant 5$, it turns out that the adjoint action of the rotation operator defining the orbifold charges $\mathcal{U}_6^{\mathbb{Z}_n} = \prod_{k=1}^{3} e^{-\frac{2\pi i}{n} Q_k \mathcal{K}_{z_k \bar{z}_k}}$ leaves invariant the following diagonal components of the metric:

$$
\begin{aligned}
K_{44} &= \frac{1}{2}(3H_2 + 4H_3 + 5H_4 + 6H_5 + 4H_6 + 2H_7 + 3H_8)\,, \\
K_{z_1\bar{z}_1} &= \frac{1}{2}(H_2 + 3H_3 + 5H_4 + 6H_5 + 4H_6 + 2H_7 + 3H_8)\,, \\
K_{z_2\bar{z}_2} &= \frac{1}{2}(H_2 + 2H_3 + 3H_4 + 5H_5 + 4H_6 + 2H_7 + 3H_8)\,, \\
K_{z_3\bar{z}_3} &= \frac{1}{2}(H_2 + 2H_3 + 3H_4 + 4H_5 + 2H_6 + H_7 + 3H_8)\,,
\end{aligned}
\tag{7.2}
$$

as well as various fields corresponding to non-zero roots:

$$
\begin{aligned}
K_{z_1\bar{z}_2}/K_{\bar{z}_1 z_2} &= \frac{1}{4}\left(E_{34} + F_{34} + E_{45} + F_{45} \pm i(E_{345} + F_{345} - (E_4 + F_4))\right)\,, \\
Z_{4z_1\bar{z}_1} &= \frac{i}{2}(E_{23^24^35^36^278} + F_{23^24^35^36^278})\,, \\
Z_{4z_2\bar{z}_2} &= \frac{i}{2}(E_{2345^26^278} + F_{2345^26^278})\,, \\
Z_{4z_3\bar{z}_3} &= \frac{i}{2}(E_{23458} + F_{23458})\,, \\
\widetilde{Z}_{z_1\bar{z}_1 z_2\bar{z}_2 z_3\bar{z}_3} &= -\frac{i}{2}(E_{34^25^36^278^2} + F_{34^25^36^278^2})\,, \\
Z_{4z_1\bar{z}_2}/Z_{4\bar{z}_1 z_2} &= \frac{1}{4}\Big(E_{23^24^25^36^278} + F_{23^24^25^36^278} + E_{234^25^26^278} + F_{234^25^26^278} \\
&\quad \pm i\big(E_{23^24^25^26^278} + F_{23^24^25^26^278} - (E_{234^25^36^278} + F_{234^25^36^278})\big)\Big)\,, \\
Z_{z_1 z_2 z_3}/Z_{\bar{z}_1\bar{z}_2\bar{z}_3} &= \frac{1}{4\sqrt{2}}\Big(E_{345^2678} + F_{345^2678} - E_{34568} - F_{34568} - E_{45^268} \\
&\quad - F_{45^268} - E_{45678} - F_{45678} \pm i\big(- E_{345^268} - F_{345^268} - E_{345678} \\
&\quad - F_{345678} - E_{45^2678} - F_{45^2678} + E_{4568} + F_{4568}\big)\Big)\,,
\end{aligned}
\tag{7.3}
$$

together with their compact counterparts, supplemented by the generators of the orbifold action $\mathcal{K}_{z_1\bar{z}_1} = -i(E_3 - F_3)$, $\mathcal{K}_{z_2\bar{z}_2} = -i(E_5 - F_5)$ and $\mathcal{K}_{z_3\bar{z}_3} = -i(E_7 - F_7)$, which bring the Cartan subalgebra to rank 7, ensuring rank conservation again.

Note that the 4 invariant combinations in the list (7.3) are in fact spanned by the elementary set of linearly independent Cartan elements satisfying $[H, E_\alpha] = 0$ for $\alpha \in \{\alpha_3, \alpha_5, \alpha_7\}$, namely: $\{2H_2 + H_3; H_3 + 2H_4 + H_5; H_5 + 2H_8; H_5 + 2H_6 + H_7\}$.

Furthermore, let us recall that the objects listed in expression (7.2) form the minimal set of invariant ladder operators for $n \geqslant 5$. In the non-generic cases $n = 3, 4$, this set is enhanced, and so is the size of $(\mathfrak{g}_{\rm inv})^{\mathbb{C}}$. We will treat these cases later on, and, for the moment being, focus on the generic invariant subalgebra for $T^6/\mathbb{Z}_{n\geqslant 5}$ only.

As before, we extract the generators corresponding to simple roots of the invariant subalgebra (the negative-root generators are omitted, since they can be obtained in a straighforward manner as $F_\alpha = (E_\alpha)^\dagger$):

$$
\begin{aligned}
E_{\tilde{\beta}} &= -iE_{2345^26^278} = (Z_{4\bar{z}_2z_2})^+ \ , \\
E_{\beta_\pm} &= \frac{1}{2}\left(E_{34} + E_{45} \pm i(E_{345} - E_4)\right) = (K_{z_1\bar{z}_2}/K_{\bar{z}_1z_2})^+ \, , \\
E_{\gamma_\pm} &= \frac{1}{2\sqrt{2}}\Big(E_{345^2678} - E_{34568} - E_{45^268} - E_{45678} \\
&\qquad \mp i\big(E_{345^268} + E_{345678} + E_{45^2678} - E_{4568}\big)\Big) = (Z_{z_1z_2z_3}/Z_{\bar{z}_1\bar{z}_2\bar{z}_3})^+ \, , \\
E_\epsilon &= -iE_{23458} = (Z_{4\bar{z}_3z_3})^+ \ ,
\end{aligned}
\tag{7.4}
$$

These generators define a complex invariant subalgebra $(\mathfrak{g}_{\rm inv})^{\mathbb{C}}$ of type $\mathfrak{d}_3 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_1 \oplus \mathbb{C}$ with the following root labeling

$$\times \quad \{E_3 - F_3 + E_5 - F_5 - 2(E_7 - F_7)\}$$

The detailed structure of the $\mathfrak{d}_3 \subset (\mathfrak{g}_{\rm inv})^{\mathbb{C}}$ is encoded in the following commutation relations:

$$
\begin{aligned}
E_{\tilde{\beta}+\beta_\pm} &\doteq \pm[E_{\tilde{\beta}}, E_{\beta_\pm}] = \frac{1}{2}\left(E_{23^24^25^36^278} + E_{234^25^26^278} \pm i(E_{23^24^25^26^278} - E_{234^25^36^278})\right) \\
&= (Z_{4z_1\bar{z}_2}/Z_{4\bar{z}_1z_2})^+ \, , \\
E_{\beta_-+\tilde{\beta}+\beta_+} &\doteq [E_{\beta_-}, E_{\tilde{\beta}+\beta_+}] = iE_{23^24^35^36^278} = (Z_{4z_1\bar{z}_1})^+ \, , \\
H_{\tilde{\beta}} &\doteq [E_{\tilde{\beta}}, F_{\tilde{\beta}}] = (H_2 + H_3 + H_4 + 2H_5 + 2H_6 + H_7 + H_8) \, , \\
H_{\beta_\pm} &\doteq [E_{\beta_\pm}, F_{\beta_\pm}] = \frac{1}{2}\left(H_3 + 2H_4 + H_5 \pm i(-E_3 + F_3 + E_5 - F_5)\right) \, .
\end{aligned}
\tag{7.5}
$$

The $\mathfrak{a}_2 \subset (\mathfrak{g}_{\rm inv})^{\mathbb{C}}$ factor is characterized as follows:

$$
\begin{aligned}
E_{\gamma_-+\gamma_+} &\doteq [E_{\gamma_-}, E_{\gamma_+}] = iE_{34^25^36^278^2} = (\widetilde{Z}_{z_1\bar{z}_1z_2\bar{z}_2z_3\bar{z}_3})^+ \, , \\
F_{\gamma_-+\gamma_+} &\doteq [F_{\gamma_+}, F_{\gamma_-}] = -iF_{34^25^36^278^2} = (\widetilde{Z}_{z_1\bar{z}_1z_2\bar{z}_2z_3\bar{z}_3})^- \, , \\
H_{\gamma_\pm} &\doteq [E_{\gamma_\pm}, F_{\gamma_\pm}] = \frac{1}{2}\left(H_3 + 2H_4 + 3H_5 + 2H_6 + H_7 + 2H_8\right) \\
&\qquad \pm\frac{i}{2}(E_3 - F_3 + E_5 - F_5 - E_7 + F_7) \, .
\end{aligned}
\tag{7.6}
$$

Finally, the Cartan generator of the remaining $\mathfrak{a}_1 \subset (\mathfrak{g}_{\rm inv})^{\mathbb{C}}$ is given by $[E_\epsilon, F_\epsilon] = H_2 + H_3 + H_4 + H_5 + H_8$. One can verify that all three simple subalgebras of $(\mathfrak{g}_{\rm inv})^{\mathbb{C}}$ indeed commute and that the compact abelian factor $i\widetilde{H}^{[6]} = (E_3 - F_3 + E_5 - F_5 - 2(E_7 - F_7))$ is the centre of $\mathfrak{g}_{\rm inv}$. The structure of $i\widetilde{H}^{[6]}$ can be retrieved from rewriting the orbifold automorphism as $\mathcal{U}_6^{\mathbb{Z}_n} = \ \exp\big((2\pi/n)\,(E_3 - F_3 + E_5 - F_5 - 2(E_7 - F_7))\big)$, and noting that it preserves $\mathfrak{g}_{\rm inv}$.

We determine the real form $\mathfrak{g}_{\rm inv}$ by a manner similar to the $T^4/\mathbb{Z}_{\mathfrak{n}\geqslant 3}$ case. Applying procedure (4.10), we find that, in a given basis, the Cartan combinations $i(H_{\beta_+} - H_{\beta_-})$ and

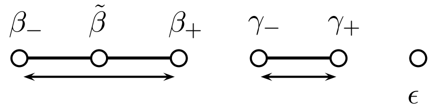
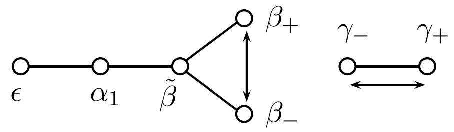
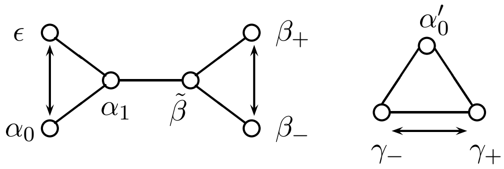
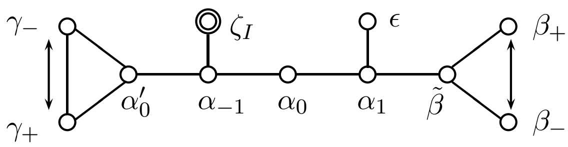
D	(Π_0, ϕ)	$\mathfrak{g}_{\rm inv}$	$\sigma(\mathfrak{g}_{\rm inv})$
4	 $\times \quad i\widetilde{H}^{[6]}$	$\mathfrak{su}(2,2) \oplus \mathfrak{su}(2,1)$ $\oplus\, \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{u}(1)$	1
3	 $\times \quad i\widetilde{H}^{[6]}$	$\mathfrak{so}(6,4) \oplus \mathfrak{su}(2,1)$ $\oplus\, \mathfrak{u}(1)$	2
2	 $\times \quad \left\{ i\widetilde{H}_n^{[6]} \right\}_{n\in\mathbb{Z}}$	$\widehat{\mathfrak{so}}(6,4) \oplus \widehat{\mathfrak{su}}(2,1)$ $\oplus\, \mathcal{L}(\mathfrak{u}(1))_{\vert -1}$	-
1	 $\times \quad i\widetilde{H}^{[6]}$	${}^6\mathcal{B}_{11(II)} \oplus \mathfrak{u}(1)$	-

Table 10: The real subalgebras $\mathfrak{g}_{\rm inv}$ for $T^{5-D} \times T^6/\mathbb{Z}_{n\geqslant 5}$ compactifications

$i(H_{\gamma_+} - H_{\gamma_-})$ are compact, resulting, for both the $\mathfrak{a}_2$ and $\mathfrak{d}_3$ subalgebras, in maximal tori $(S^1)\oplus\mathbb{R}^{\oplus^{r-1}}$, for $r = 2, 3$, respectively. Taking into account the remaining $\mathfrak{u}(1)$ factor, it is easy to see that $\mathfrak{g}_{\rm inv} = \mathfrak{su}(2,2)\oplus\mathfrak{su}(2,1)\oplus\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{u}(1)$, with overall signature $\sigma(\mathfrak{g}_{\rm inv}) = 1$.

This and further compactifications of the theory are listed in Table 10. In $D = 3$ the roots ϵ and $\tilde{\beta}$ listed in expression (7.4) connect through α_1, producing the invariant real form $\mathfrak{g}_{\rm inv} = \mathfrak{so}(6,4) \oplus \mathfrak{su}(2,1) \oplus \mathfrak{u}(1)$.

In $D = 2$, the invariant subalgebra is now a triple affine central product $\hat{\mathfrak{d}}_5 \bowtie \hat{\mathfrak{a}}_2 \bowtie \hat{\mathfrak{u}}(1) \equiv \hat{\mathfrak{d}}_5 \bowtie (\hat{\mathfrak{a}}_2 \bowtie \hat{\mathfrak{u}}(1))$, associatively. For convenience, we have once again depicted in Table 10 the direct sum before identification of centres and scaling operators. When carrying out the identification, the affine root α'_0 has thus to be understood as a non-simple root in $\Delta_+(E_9)$, enforcing: $\alpha'_0 = \delta - (\gamma_+ + \gamma_-)$. Moreover, it can be checked that $\delta_{D_5} \doteq \alpha_0 + \epsilon + 2(\alpha_1 + \tilde{\beta}) + \beta_+ + \beta_- = \delta$, resulting in $H_{\delta_{D_5}} = H_{\delta_{A_2}} = c_{\hat{\mathfrak{u}}(1)}$, for $\delta_{A_2} = \alpha'_0 + \gamma_+ + \gamma_-$. This identification carries over to the three corresponding scaling operators.

The reality properties of $\mathfrak{g}_{\rm inv}$ can be inferred from the finite case, by extending the analysis of the $T^5 \times T^4/\mathbb{Z}_{n\geqslant n}$ orbifold in Section 6.3 separately to the $\hat{\mathfrak{d}}_5$ and $\hat{\mathfrak{a}}_2$ factors.

Since both subalgebras are "next to split", the $\hat{\mathfrak{d}}_5$ case is directly retrievable from the construction exposed in Section 6.3 by reducing the rank by two. The real $\widehat{\mathfrak{su}}(2,1)$ factor is also characterized by an automorphism $U(z)$ of type 1a, with $u = 1$, which can be found, along with further specifications, in [70]. The rest of the analysis is similar to the discussion for the $T^5 \times T^4/\mathbb{Z}_{n\geqslant n}$ case. The signatures $\hat{\sigma}$ of both non-abelian factors are infinite again, and can be decomposed as in expression (6.35).

In $D = 1$, the Borcherds algebra ${}^6\mathcal{B}_{11}$ resulting from reconnecting the three affine KMAs appearing in $D = 2$ through the extended root α_{-1} is defined this time by a Cartan matrix of corank 2, with simple imaginary root ζ_I attached to the raising operator $E_{\zeta_I} = (1/2)(E_{\delta+\alpha_3} - E_{\delta-\alpha_3} + E_{\delta+\alpha_5} - E_{\delta-\alpha_5} - 2E_{\delta+\alpha_7} + 2E_{\delta-\alpha_5})$. Since the triple extension is not successive, the ensuing algebra is more involved than an EALA. Writing for short $\delta_2 \doteq \delta_{A_2} = \alpha_0' + \gamma_+ + \gamma_-$, it possesses two centres, namely $\{\mathfrak{z}_1 = H_\delta - H_I, \mathfrak{z}_2 = H_{\delta_2} - H_\delta\}$ and two scaling elements $\{d_I, d_2\}$ counting the levels in ζ_I and δ_2. The signature $\hat{\sigma}$ of $\mathfrak{g}_{\rm inv}$ is again undefined.

Denoting by ${}^6\mathcal{B}_{11(II)}$ the real Borcherds algebra represented in Table 10, the II referring to the two arrows connecting respectively $\gamma_\pm$ and $\beta_\pm$ in the Satake diagram, the real form $\mathfrak{g}_{\rm inv}$ is given by

$$\mathfrak{g}_{\rm inv} = \mathfrak{u}(1) \oplus {}^6\mathcal{B}_{11(II)}/\{\mathfrak{z}_1; \mathfrak{z}_2; d_I; d_2\}\,.$$

By construction, $-H_{-1}$ will replace d_I and d_2 after the quotient is performed.

7.2 The non-generic $n = 4$ case

As we mentioned at the beginning of this section, there is a large number of consistent choices for the charges of the $T^6/\mathbb{Z}_n$ orbifold. Moreover, non-generic invariant subalgebras appear for particular periodicities n. For our choice of orbifold charges, the non-generic cases appear in $n = 3, 4$, and are singled out from the generic one by the absence of a $\mathfrak{u}(1)$ factor in the invariant subalgebra. In $D = 1$, this entails the appearance of simple invariant Kac-Moody subalgebras in place of the simple Borcherds type ones encountered up to now. These KMA will be denoted by $\mathcal{KM}$.

The novelty peculiar to the $T^6/\mathbb{Z}_4$ orbifold lies in the invariance of the root α_7, which is untouched by the mirror symmetry $(z_3, \bar{z}_3) \to (-z_3, -\bar{z}_3)$, so that the generators E_7, F_7 and H_7 are now conserved separately. Furthermore, several new invariant generators appear related to $Z_{z_1 z_2 \bar{z}_3}$ and $Z_{\bar{z}_1 \bar{z}_2 z_3}$:

$$\begin{aligned}
&Z_{z_1 z_2 \bar{z}_3}/Z_{\bar{z}_1 \bar{z}_2 z_3} = \\
&\quad \frac{1}{4\sqrt{2}}\big(E_{345^2678} + F_{345^2678} + E_{34568} + F_{34568} + E_{45^268} + F_{45^268} - E_{45678} - F_{45678} \\
&\quad \pm i(E_{345^268} + F_{345^268} - E_{345678} - F_{345678} - E_{45^2678} - F_{45^2678} - E_{4568} - F_{4568})\big)\,,
\end{aligned}$$

together with the corresponding compact generators.

The invariant subalgebra is now more readily derived by splitting the Z generators into combinations containing or not an overall AdE_7 factor (in other words, we "decomplexify"

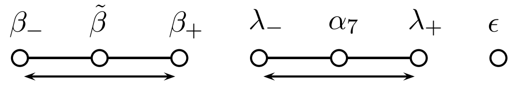
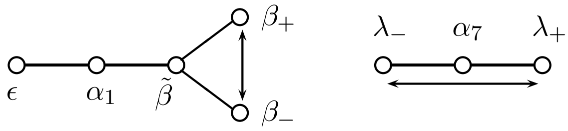
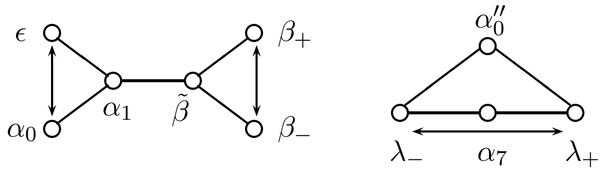
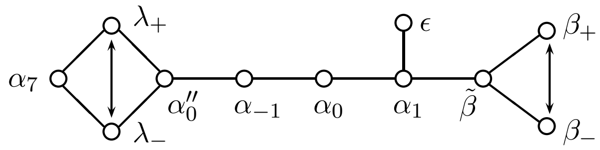
D	(Π_0, ϕ)	$\mathfrak{g}_{\text{inv}}$	$\sigma(\mathfrak{g}_{\text{inv}})$
4		$\mathfrak{su}(2,2)^{\oplus^2} \oplus \mathfrak{sl}(2,\mathbb{R})$	3
3		$\mathfrak{so}(6,4) \oplus \mathfrak{su}(2,2)$	4
2		$\widehat{\mathfrak{so}}(6,4) \oplus \widehat{\mathfrak{su}}(2,2)$	-
1		${}^6\mathcal{KM}_{11(II)}$	-

Table 11: The real subalgebras $\mathfrak{g}_{\text{inv}}$ for $T^{5-D} \times T^6/\mathbb{Z}_4$ compactifications

z_3 into x^9 and x^{10}):

$$\begin{aligned} E_{\lambda_\pm} &= -\frac{i}{2}\big(E_{34568} + E_{45^268} \pm i(E_{345^268} - E_{4568})\big) = (Z_{z_1 z_2 10}/ - Z_{\bar z_1 \bar z_2 10})^+ \, , \\ E_{\alpha_7+\lambda_\pm} &= \frac{1}{2}\big(E_{345^2678} - E_{45678} \pm i(E_{345678} + E_{45^2678})\big) = (Z_{\bar z_1 \bar z_2 9}/Z_{z_1 z_2 9})^+ \end{aligned} \tag{7.7}$$

which verify the following algebra:

$$\begin{aligned} E_{\alpha_7+\lambda_\pm} &\doteq \pm[E_{\alpha_7}, E_{\lambda_\pm}] \, , \\ E_{\lambda_-+\alpha_7+\lambda_+} &\doteq [E_{\lambda_-}, E_{\alpha_7+\lambda_+}] = -E_{34^25^36^278^2} = -i(\widetilde{K}_{z_1\bar z_1 z_2 \bar z_2 z_3 \bar z_3})^+, \\ H_{\lambda_\pm} &\doteq [E_{\lambda_\pm}, F_{\lambda_\pm}] = \frac{1}{2}\left(H_3 + 2H_4 + 3H_5 + 2H_6 + 2H_8 \pm i(E_3 - F_3 + E_5 - F_5)\right) \, , \end{aligned} \tag{7.8}$$

so that the former $\mathfrak{a}_2$ factor for n generic is now enhanced to $\mathfrak{a}_3$. One compact combination $i(H_{\lambda_+} - H_{\lambda_-})$ results from the action of $\text{Fix}_{\mathcal{T}_0}$ on the algebra formed by the generators in (7.8), which determines the corresponding real form to be $\mathfrak{su}(2,2)$. The chain of invariant subalgebras resulting from further compactifications follows as summarized in Table 11.

In $D = 2$, we have the identification $\alpha_0'' = \delta - (\lambda_+ + \alpha_7 + \lambda_-)$ leading to the now customary affine central product $\widehat{\mathfrak{so}}(6,4) \bowtie \widehat{\mathfrak{su}}(2,2)$, represented for commodity as a direct sum in Table 11. The corresponding Satake diagram defines the real form $\mathfrak{g}_{\rm inv}$. Its signature $\hat{\sigma}$ is infinite with the correspondence $\hat{\sigma}(\mathfrak{g}_{\rm inv}|_{D=2}) = 2 + 2 \times 4 \times \infty = \sigma(\mathfrak{g}_{\rm inv}|_{D=3}) - 2 + 2 \times \sigma(\mathfrak{g}_{\rm inv}|_{D=3}) \times \infty$.

In $D = 1$, $\mathfrak{g}_{\rm inv}$ is defined by the quotient of a simple KMA: ${}^6\mathcal{KM}_{11}$, by its centre $\mathfrak{z} = H_\delta - H_{\delta_3}$, where $\delta_3 = \alpha_0'' + \lambda_+ + \alpha_7 + \lambda_-$, and by the derivation d_3. As for affine KMAs, ${}^6\mathcal{KM}_{11}$ is characterized by a degenerate Cartan matrix with rank $r = 2 \times 11 - \dim \mathfrak{h} = 10$. However, the principal minors of its Cartan matrix are not all strictly positive, so that ${}^6\mathcal{KM}_{11}$ does not result from the standard affine extension of any finite Lie algebra. The real form $\mathfrak{g}_{\rm inv}$ is determined from the Satake diagram in Table 11 and the relation:

$$\mathfrak{g}_{\rm inv} = {}^6\mathcal{KM}_{11(II)}/\{\mathfrak{z}; d_3\}\,. \tag{7.9}$$

Our convention denotes by II the class of real forms of ${}^6\mathcal{KM}_{11}$ for which the Cartan involution exhibits both possible diagram symmetries, exchanging $\phi(\lambda_\pm) = \mp\lambda_\pm$ and $\phi(\beta_\pm) = \mp\beta_\pm$.

7.3 The standard $n = 3$ case

Starting in $D = 4$, the $\mathbb{Z}_3$-invariant subalgebra builds up the semi-simple $(\mathfrak{g}_{\rm inv})^{\mathbb{C}} = \mathfrak{a}_5 \oplus \mathfrak{a}_2$. The $\mathfrak{a}_5$ part follows from enhancing the $\mathfrak{a}_3 \oplus \mathfrak{a}_1$ semi-simple factor of the generic invariant subalgebra (7.4) by the following additional invariant generators:

$$\begin{aligned}
K_{z_1\bar{z}_3}/K_{\bar{z}_1 z_3} &= \frac{1}{4}\big(E_{3456} + F_{3456} + E_{4567} + F_{4567} \\
&\qquad \pm i(E_{34567} + F_{34567} - E_{456} - F_{456})\big)\,, \\
K_{z_2\bar{z}_3}/K_{\bar{z}_2 z_3} &= \frac{1}{4}\big(E_{56} + F_{56} + E_{67} + F_{67} \pm i(E_{567} + F_{567} - E_6 - F_6)\big)\,, \\
Z_{4 z_1\bar{z}_3}/Z_{4\bar{z}_1 z_3} &= \frac{1}{4}\big(E_{23^24^25^2678} + F_{23^24^25^2678} + E_{234^25^268} + F_{234^25^268} \\
&\qquad \pm i(E_{23^24^25^268} + F_{23^24^25^268} - E_{234^25^2678} - F_{234^25^2678})\big)\,, \\
Z_{4 z_2\bar{z}_3}/Z_{4\bar{z}_2 z_3} &= \frac{1}{4}\big(E_{2345^2678} + F_{2345^2678} + E_{234568} + F_{234568} \\
&\qquad \pm i(E_{2345^268} + F_{2345^268} - E_{2345678} - F_{2345678})\big)\,,
\end{aligned} \tag{7.10}$$

(together with their corresponding compact generators). It becomes clear that one is

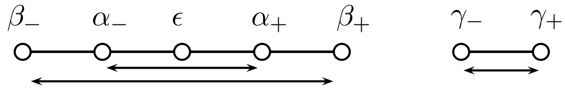
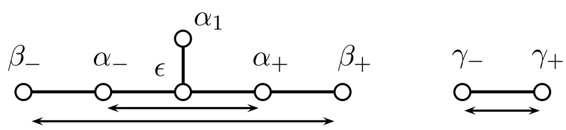
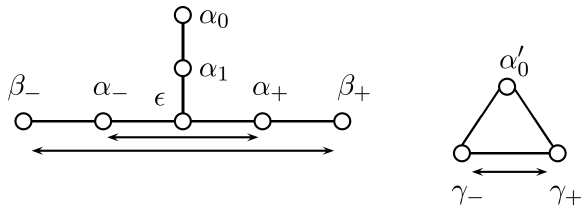
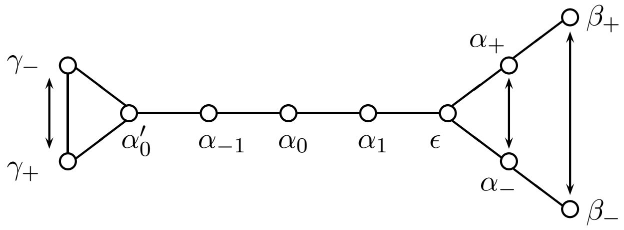
D	(Π_0,ϕ)	$\mathfrak{g}_{\rm inv}$	$\sigma(\mathfrak{g}_{\rm inv})$
4		$\mathfrak{su}(3,3)\oplus\mathfrak{su}(2,1)$	1
3		$\mathfrak{e}_{6\|2}\oplus\mathfrak{su}(2,1)$	2
2		$\hat{\mathfrak{e}}_{6\|2}\oplus\widehat{\mathfrak{su}}(2,1)$	-
1		${}^{6'}\mathcal{KM}_{11(III)}$	-

Table 12: The real subalgebras $\mathfrak{g}_{\rm inv}$ for $T^{5-D}\times T^6/\mathbb{Z}_3$ compactifications

dealing with an $\mathfrak{a}_5$-type algebra, when recasting the whole system in the basis:

$$\begin{aligned}
E_\epsilon &\doteq [E_2,[E_3,E_{\tilde\alpha}]] = -iE_{23458} = (Z_{4z_3\bar z_3})^+\,,\\
E_{\alpha_\pm} &\doteq \frac{1}{2}\left(E_{56}+E_{67}\pm i(E_{567}-E_6)\right) = (K_{z_2\bar z_3}/K_{\bar z_2 z_3})^+\,,\\
E_{\beta_\pm} &\doteq \frac{1}{2}\left(E_{34}+E_{45}\pm i(E_{345}-E_4)\right) = (K_{z_1\bar z_2}/K_{\bar z_1 z_2})^+\,,\\
E_{\alpha_\pm+\beta_\pm} &\doteq \frac{1}{2}\left(E_{3456}+E_{4567}\pm i(E_{34567}-E_{456})\right) = (K_{z_2\bar z_3}/K_{\bar z_2 z_3})^+\,,\\
E_{\epsilon+\alpha_\pm} &\doteq \frac{1}{2}\left(E_{2345^2678}+E_{234568}\pm i(E_{2345^268}-E_{2345678})\right)\\
&\quad = (Z_{4z_2\bar z_3}/Z_{4\bar z_2 z_3})^+\,, \qquad (7.11)\\
E_{\epsilon+\alpha_\pm+\beta_\pm} &\doteq \frac{1}{2}\left(E_{23^24^25^2678}+E_{234^25^268}\pm i(E_{23^24^25^268}-E_{234^25^2678})\right)\\
&= (Z_{4z_1\bar z_3}/Z_{4\bar z_1 z_3})^+\,,\\
E_{\alpha_-+\alpha_{23458}+\alpha_+} &\doteq iE_{2345^26^278} = (Z_{4z_2\bar z_2})^+\,,\\
E_{\alpha_-+\epsilon+\alpha_++\beta_\pm} &\doteq \frac{1}{2}\left(E_{23^24^25^36^278}+E_{234^25^26^278}\pm i(E_{23^24^25^26^278}-E_{234^25^36^278})\right)\\
&= (Z_{4z_1\bar z_2}/Z_{4\bar z_1 z_2})^+\\
E_{\beta_-+\alpha_-+\epsilon+\alpha_++\beta_+} &\doteq iE_{23^24^35^36^278} = (Z_{4z_1\bar z_1})^+\,,
\end{aligned}$$

Combining expressions (7.5) and (7.11) reproduces the commutation relations of an $\mathfrak{a}_5$-type algebra. The remaining factor $\mathfrak{a}_3 \subset (\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$ is kept untouched from the generic case. Choosing an appropriate basis for the Cartan subalgebra of $\mathfrak{g}_{\text{inv}}$ produces three compact combinations $i(H_{\gamma_+} - H_{\gamma_-})$, $i(H_{\alpha_+} - H_{\alpha_-})$ and $i(H_{\beta_+} - H_{\beta_-})$, leaving four non-compact ones, which determines $\mathfrak{g}_{\text{inv}} = \mathfrak{su}(3,3) \oplus \mathfrak{su}(2,1)$.

The $T^{5-D} \times T^6/\mathbb{Z}_3$ chain of real invariant subalgebras follows as depicted in Table 12, for $D = 4, \dots, 1$. In $D = 2$, as for the $n = 4$ case, we can associate a formal signature $\hat{\sigma}$ to the real form $\mathfrak{g}_{\text{inv}} = \hat{\mathfrak{e}}_{6|2} \bowtie \widehat{\mathfrak{su}}(2,1)$, which is infinite but keeps a trace of the $D = 3$ finite case for which $\sigma(\mathfrak{g}_{\text{inv}}) = 2$, as $\hat{\sigma}(\mathfrak{g}_{\text{inv}}|_{D=2}) = \sigma(\mathfrak{g}_{\text{inv}}|_{D=3}) - 2 + 2 \times \sigma(\mathfrak{g}_{\text{inv}}|_{D=3}) \times \infty = 0 + 2 \times 2 \times \infty$. In $D = 1$, the III subscript labeling the real form appearing in Table 12 refers, as before, to the number of arrows in its defining Satake diagram, and the real invariant subalgebra is retrieved from modding out ${}^{6'}\mathcal{KM}_{11(III)}$ in a way similar, modulo the required changes, to expression (7.9).

8. Non-linear realization of the $\mathbb{Z}_n$-invariant sector of M-theory

In this section, we want to address one last issue concerning the invariant (untwisted) sector of these orbifolds, namely how the residual symmetry $\mathfrak{g}_{\text{inv}}$ can be made manifest in the equations of motion of the orbifolded supergravity in the finite-dimensional case, and in the effective $D = 1$ σ-model description of M-theory near a space-time singularity in the infinite-dimensional case. The procedure follows the theory of non-linear σ-models realization of physical theories from coset spaces, more particularly from the conjectured effective Hamiltonian on $E_{10|10}/K(E_{10|10})$ presented in Sections 3.2 and 3.4.

It is customary in this context to choose the Borel gauge to fix the class representatives in the coset space. To do this in our orbifolded case, we need the Iwasawa decomposition of the real residual U-duality algebra $\mathfrak{g}_{\text{inv}}$, which can be deduced from its restricted-root space decomposition (see Section 4.3). For this purpose, we build the set of restricted roots Σ_0 (4.12) for $\mathfrak{g}_{\text{inv}}$ and partition it into a set of positive and a set of negative restricted roots $\Sigma_0 = \Sigma_0^+ \cup \Sigma_0^-$, based on some lexicographical ordering. Then, following definition (4.11), we build:

$$\mathfrak{n}_{\text{inv}}^{(\pm)} = \bigoplus_{\bar{\alpha} \in \Sigma_0^{\pm}} (\mathfrak{g}_{\text{inv}})_{\bar{\alpha}} \,, \qquad \text{with } \phi(\mathfrak{n}_{\text{inv}}^{(\pm)}) = \mathfrak{n}_{\text{inv}}^{(\mp)} \,. \tag{8.1}$$

Identifying the nilpotent algebra as $\mathfrak{n}_{\text{inv}} \equiv \mathfrak{n}_{\text{inv}}^{(+)}$, the Iwasawa decomposition of $\mathfrak{g}_{\text{inv}}$ is given by $\mathfrak{g}_{\text{inv}} = \mathfrak{k}_{\text{inv}} \oplus \mathfrak{a}_{\text{inv}} \oplus \mathfrak{n}_{\text{inv}}$, and the coset parametrization in the Borel gauge by $\mathfrak{g}_{\text{inv}}/\mathfrak{k}_{\text{inv}} = \mathfrak{a}_{\text{inv}} \oplus \mathfrak{n}_{\text{inv}}$.

The cases where $\mathfrak{g}_{\text{inv}}$ is a finite real Lie algebra are easily handled. For $8 \leqslant D \leqslant 3$, the Satake diagrams listed in Tables 3, 9, 10, 11 and 12 describing the residual U-duality algebras under $T^q/\mathbb{Z}_n$ orbifolds for $q = 2, 4, 6$ and $n \geqslant 3$ are well known, and the corresponding Dynkin diagrams for the basis $\overline{\Pi}_0$ of restricted roots Σ_0 can be found, together with the associated multiplicities $m_r(\bar{\alpha}_i)$, in [75]. We will not dwell on the $D = 2$ case, which only serves as a stepping stone to the understanding of the $D = 1$ case. Moreover, all the arguments we present here regarding $D = 1$ also apply, with suitable restrictions, to the $D = 2$ case.

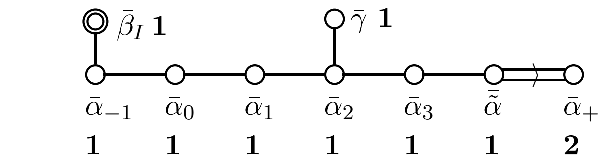
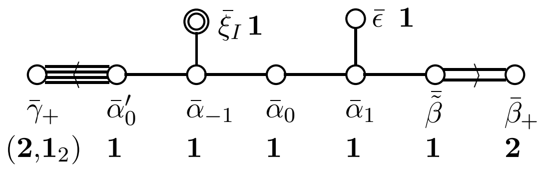
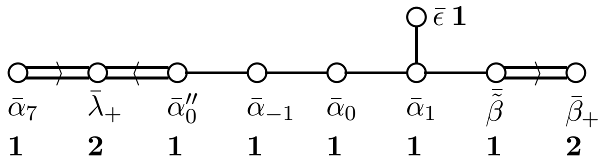
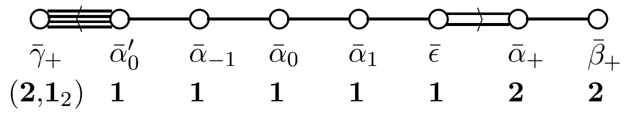
	$\overline{\Pi}_0$
$T^6 \times T^4/\mathbb{Z}_{n>2}$: ${}^4\mathcal{B}_{10(Ib)}$	
$T^4 \times T^6/\mathbb{Z}_{n\geq 5}$: ${}^6\mathcal{B}_{11(II)}$	
$T^4 \times T^6/\mathbb{Z}_4$: ${}^6\mathcal{KM}_{11(II)}$	
$T^4 \times T^6/\mathbb{Z}_3$: ${}^{6'}\mathcal{KM}_{11(III)}$	

Table 13: Restricted Dynkin diagrams for very-extended $\mathfrak{g}_{\rm inv}$ subalgebras

In the $D = 1$ case, since the restricted root space is best determined from the Satake diagram of the real form, we will replace $\mathfrak{g}_{\rm inv}$ in eqn. (8.1) and all ensuing formulae by the real Borcherds and indefinite KM algebras described by the corresponding Satake diagrams in Tables 3 and 9-12. This procedure leads to the Dynkin diagrams and root multiplicities represented in Table 13 for the bases of restricted roots $\overline{\Pi}_0$, given for all but the split case of Table 3 (for which normal and restricted roots coincide). The multiplicities appear in bold beside the corresponding restricted root, while we denote the multiplicity of $2\bar{\alpha}$ with a 2 subscript whenever it is also a root of Σ_0.

We may now give a general prescription to compute the algebraic field strength $\mathcal{G} = g^{-1}dg$ of the orbifolded theory, which also applies to the infinite-dimensional case, where $\mathcal{G}$ is the formal coset element (3.45). There are two possible equivalent approaches, depending on whether we set to zero the dual field associated to the possible centres and derivations of the Borcherds/KMA algebras leading to $\mathfrak{g}_{\rm inv}$ before or after the computation of $\mathcal{G}$.

Let us consider an algebra with A pairs of non-compact centres and derivations: $\{\mathfrak{z}_a, d_a\}_{a=1,\ldots,A}$ and a maximal non-compact abelian subalgebra $\mathfrak{a}$ of dimension n_s. We introduce a vector of $n_s - 2A$ scale factors $\bar{\varphi}$ and a vector of auxiliary fields ψ and develop them on the basis of $\mathfrak{a}(\mathfrak{g})$ as:

$$\bar{\varphi} = \sum_{\bar{\imath}=1}^{n_s-2A} \bar{\varphi}^i H_{\bar{\imath}}\,, \qquad \psi = \sum_{a=1}^{A} \left(\psi^a \mathfrak{z}_a + \psi^{A+a} d_a\right)\,. \tag{8.2}$$

For example, in the case of the $T^6 \times T^4/\mathbb{Z}_n$ orbifold, $\mathfrak{a}_{\mathrm{inv}}$ is given by:

$$\begin{aligned}\mathfrak{a}({}^4\mathcal{B}_{10(Ib)}) = \mathrm{Span}_{\mathbb{R}}\{&H_{\bar{\alpha}_{-1}};\ldots;H_{\bar{\alpha}_3};H_{\bar{\bar{\alpha}}} = H_4 + H_5 + H_8; H_{\bar{\alpha}_+} = H_5 + 2H_6 + H_7;\\ &H_{\bar{\gamma}} = H_2 + 2H_3 + 3H_4 + 3H_5 + 2H_6 + H_7 + H_8; \mathfrak{z}; d_I\}\,.\end{aligned}$$

in which the centre is $\mathfrak{z} = H_{\bar{\beta}_I} - (H_{\bar{\alpha}_{-1}} + \ldots + H_{\bar{\alpha}_3} + H_{\bar{\bar{\alpha}}} + H_{\bar{\alpha}_+} + H_{\bar{\gamma}}) = H_{\beta_I} - H_\delta$. The generator $H_{\bar{\imath}}$ will be understood to represent the i-th element of the above list, for $\bar{\imath} = 1, \ldots, 8$.

In general, a central element obviously does not contribute much to $\mathcal{G}$, except for a term $\propto d\psi^a/dt$, so that it does not matter whether we impose the physical constraint $\psi^a = 0$, $\forall a = 1, \ldots, A$, before or after the computation of $\mathcal{G}$. The derivations d_a also create terms $\propto d\psi^{A+a}/dt$, but ψ^{A+a} appears in exponentials in front of generators for roots containing an imaginary/affine ($\neq \alpha_0$) simple root, as well. However, there is no difference between setting the auxiliary field ψ to zero directly in g or, later, in the exponentials in $\mathcal{G}$. Indeed, the counting of levels in these imaginary/affine simple roots is taken care of by $H_{\bar{\alpha}_{-1}}$ anyway. Finally, a term proportional to $\mathfrak{z}_a$ other than $\mathfrak{z}_a\, d\psi^a/dt$ can not be produced either, since we work in the Borel gauge, so that terms containing commutators like $[E_{\beta_I}, F_{\beta_I}]$ are absent. It is thus not necessary to impose $\mathfrak{z}_a = 0$ $\forall a$ again in the end.

More generally, let us now set $n_{\mathrm{inv}} = \bigoplus_{\bar{\alpha}\in\Sigma_0^+} (\mathfrak{g}_{\mathrm{inv}})_{\bar{\alpha}}$ with $\dim(\mathfrak{g}_{\mathrm{inv}})_{\bar{\alpha}} = m(\bar{\alpha}) \cdot m_r(\bar{\alpha})$, the (formal) group element[13]

$$g = e^{\bar{\varphi}+\psi} \cdot \prod_{\bar{\alpha}\in\Sigma_0^+} \prod_{s=1}^{m_r(\bar{\alpha})} \prod_{a=1}^{m(\bar{\alpha})} e^{C_{\bar{\alpha},(s,a)} E_{\bar{\alpha}}^{(k,a)}}\,, \tag{8.3}$$

[13]Note that we adopt here a perspective that is different from [31] when associating restricted roots to the metric and p-form potential of orbifolded $11D$ supergravity / M-theory. In particular, the authors of [32] were concerned with super-Borcherds symmetries of supergravity with non-split U-duality groups, which form the so-called real magic triangle, i.e. which are oxidations of pure supergravity in 4 dimensions with $\mathcal{N} = 0, .., 7$ supersymmetries. When doubling the fields of these theories by systematically introducing Hodge duals for all original p-form fields (but not for the metric), the duality symmetry of this enlarged model can be embedded in a larger Borcherds superalgebra. The self-duality equations for all p-forms of these supergravities can be recovered by a certain choice of truncation in the Grassmanian coefficients of the superalgebra. In contrast to our approach however, one positive restricted root was related to one p-form potential term in [32], whereas, we associate a restricted root generator to one component of the potential. This is the reason why these authors drop the sum over $m_r(\bar{\alpha})$ in expression (8.3) (not mentioning the sum over $m(\bar{\alpha})$ in the $D = 1$ case, which we keep since we do not want to discard any higher order contributions to classical 11D supergravity)

can be used to compute the Maurer-Cartan one-form:

$$\mathcal{G} = \left[g^{-1}dg\right]_{\psi^a=0,a=1,\dots,2A}$$

in which the coefficients $\{\bar{\varphi}^i; C_{\bar{\alpha}}\}$ correspond, at levels $l = 0, 1$, to the invariant dilatons and potentials of orbifolded classical 11D supergravity, and, at higher levels, participate to (invariant) contributions from M-theory. Their exact expressions can be reconstructed from the material of Sections 5–7 and the Satake diagrams of Tables 3 and 9–12.

The Maurer-Cartan equation $d\mathcal{G} = \mathcal{G} \wedge \mathcal{G}$ will then reproduce the equations of motion of the untwisted sector of the reduced supergravity theory in the finite case, of M-theory on $M = \mathbb{R}_+ \times T^{11-D-2p} \times T^{2p}/\mathbb{Z}_n$ in $D = 1$, which will make manifest the residual symmetry $\mathfrak{g}_{\rm inv}$. Finally, one can write down an effective invariant Hamiltonian as in expression (3.44), by performing a Legendre transform of $\mathcal{L} = \frac{1}{4n}\left[\mathrm{Tr}(\partial\mathcal{M}^{-1}\partial\mathcal{M})\right]_{\psi^a=0,a=1,\dots,2A}$, the orbifolded version of expression (3.25).

9. Shift vectors and chief inner automorphisms

We have dedicated the first few chapters of this article to explaining the characterization of fixed-point subalgebras under finite-order automorphisms of U-duality algebras. In physical words, we have computed the residual U-duality symmetry of maximally supersymmetric supergravities compactified on certain toric orbifolds. In $D \geq 3$, the quotient of this residual algebra by its maximal compact subalgebra is in one-to-one correspondence with the physical spectrum of $11D$ supergravity surviving the orbifold projection. In string theory language, this corresponds to the untwisted sector of the orbifolded theory. Extrapolating this picture to $D = 1$, the orbifold spectrum gets enhanced by a whole tower of massive string states and/or non-perturbative states.

Although the interpretation of most of these higher level $\mathfrak{e}_{10}$ roots is still in its infancy, an interesting proposal was made in [19] for a restricted number of them, namely for those appearing as shift vectors describing $\mathbb{Z}_2$ orbifold actions. They were interpreted as the extended objects needed for local anomaly and charge cancellations in brane models of certain M-theory orbifolds and type IIA orientifolds.

In this section, a general method to compute the shift vectors of any $T^p \times T^q/\mathbb{Z}_n$ orbifolds will be given, as well as explicit results for $q = 2, 4, 6$. Then, an empirical technique to obtain $\mathfrak{e}_{10}$ roots that are physically interpretable will be presented, exploiting the freedom in choosing a shift vector from its equivalence class. Our results in particular reproduce the one given in [19] for $T^6 \times T^4/\mathbb{Z}_2$. Note that the method will allow to differentiate, for example, between $T^4/\mathbb{Z}_3$ and $T^4/\mathbb{Z}_4$ despite the fact that they lead to the same invariant subalgebra, which gives a clue on the rôle of the n-dependent part of the shift vectors. Finally, we will see how to extract the roots describing a $\mathbb{Z}_n$ orbifolds from all level $3n$ roots of $\mathfrak{e}_{10}$.

We first remark that the complex combinations of generators corresponding to the complexified physical fields are the eigenvectors of the automorphisms $\mathcal{U}_q^{\mathbb{Z}_n}$ with eigenvalues $\exp(i\frac{2\pi}{n}Q_A)$, for $Q_A \in \mathbb{Z}_n$. Having a basis of eigenvectors suggests that there is a conjugate Cartan subalgebra $\mathfrak{h}'$ inside of the $Q_A = 0$ eigenspace $\mathfrak{g}^{(0)}$ for which the automorphism is diagonal. We can then reexpress the orbifold action as an automorphism that leaves this new Cartan subalgebra invariant, in other words as a chief inner automorphism of the form $\mathrm{Ad}(\exp(i\frac{2\pi}{n}H'))$ for some $H' \in \mathfrak{h}'_{\mathbb{Q}}$. As already noticed in the case of $\mathbb{Z}_2$ orbifolds, such a chief inner automorphism simply acts as $\exp(i\frac{2\pi}{n}\alpha'(H'))$ on every root subspace $\mathfrak{g}_{\alpha'}$ where α' is a root defined with respect to $\mathfrak{h}'$. In particular, we can find a (non-unique) weight vector Λ' corresponding to H' so that

$$\mathrm{Ad}(e^{i\frac{2\pi}{n}H'})\mathfrak{g}_{\alpha'} = e^{i\frac{2\pi}{n}(\Lambda'|\alpha')}\mathfrak{g}_{\alpha'}\,.$$

Such a weight vector is commonly called shift vector. It turns out that all automorphisms of a given simple Lie algebra can be classified by all weights $\Lambda' = \sum_{i=1}^{r} l_i \Lambda'^i$, $l_i \in \mathbb{Z}_n$ without common prime factor, so that, according to [33],

$$(\Lambda|\theta_G) \leq n \tag{9.1}$$

(here, the fundamental weights Λ'^i are defined to be dual to the new simple roots α'_i, i.e. $(\Lambda'^i|\alpha'_j) = \delta^i_j\ \forall i,j = 9-r,\ldots,8$). Furthermore, there is a simple way, see [29], to deduce the invariant subalgebra from Λ by guessing its action on the extended Dynkin diagram, if Λ satisfies the above condition. Here, we will first show how to obtain the shift vectors in the cases we are interested in and then describe the above-mentioned diagrammatic method with the help of these examples.

9.1 A class of shift vectors for $T^2/\mathbb{Z}_n$ orbifolds

Let us start by the particularly simple case of a $T^2/\mathbb{Z}_n$ orbifold in T^3. We can directly use the decomposition in eigensubspaces obtained in equation (2). The first task is to choose a new Cartan subalgebra, or equivalently a convenient Cartan-Weyl basis. In other words, we are looking for a new set of simple roots for $\mathfrak{a}_2 \oplus \mathfrak{a}_1$, so that all Cartan generators are in the $Q_A = 0$ eigensubspace $\mathfrak{g}^{(0)}$. Since the $\mathfrak{a}_1$ does not feel the orbifold action, we can simply take $H'_8 = H_8$. On the other hand, we should take for H'_6 and H'_7 some combinations of $2H_6 + H_7$ and $E_7 - F_7$. A particularly convenient choice is given by the following Cartan-Weyl basis:

$$\begin{aligned}
&E'_6 = \tfrac{1}{\sqrt{2}}(E_6 + iE_{67})\,, && F'_6 = \tfrac{1}{\sqrt{2}}(F_6 - iF_{67})\,, && H'_6 = \tfrac{1}{2}(2H_6 + H_7 - i(E_7 - F_7))\,,\\
&E'_7 = \tfrac{1}{2}(H_7 - i(E_7 + F_7))\,, && F'_7 = \tfrac{1}{2}(H_7 + i(E_7 + F_7))\,, && H'_7 = i(E_7 - F_7)\,,\\
&E'_{67} = \tfrac{1}{\sqrt{2}}(E_6 - iE_{67})\,, && F'_{67} = \tfrac{1}{\sqrt{2}}(F_6 + iF_{67})\,, && H'_{67} = \tfrac{1}{2}(2H_6 + H_7 + i(E_7 - F_7))\,,
\end{aligned}$$

This gives the following simple decomposition in eigensubspaces:

$$\begin{aligned}
&\mathfrak{g}^{(0)} = \mathrm{Span}\{H'_6; H'_7; E'_8; F'_8; H'_8\}\,, && \mathfrak{g}^{(n-1)} = \mathrm{Span}\{E'_{67}; F'_6\}\,,\\
&\mathfrak{g}^{(1)} = \mathrm{Span}\{E'_6; F'_{67}\}\,, && \mathfrak{g}^{(n-2)} = \mathrm{Span}\{E'_7\}\,,\\
&\mathfrak{g}^{(2)} = \mathrm{Span}\{F'_7\}\,.
\end{aligned}$$

Notice that $\mathfrak{g}^{(n-i)}$ is obtained from $\mathfrak{g}^{(i)}$ by the substitution $E \leftrightarrow F$, so that we will only give the latter explicitly in the following examples. Furthermore, since $\mathcal{U}_2^{\mathbb{Z}_n}$ actually defines a gradation $\mathfrak{g} = \bigoplus_{i=0}^{n-1}\mathfrak{g}^{(i)}$, we have the property $[\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] \subseteq \mathfrak{g}^{(i+j)}$. This implies that if we can find a weight $\Lambda'^{\{2\}}$ that acts as $\exp(i\frac{2\pi}{n}(\Lambda'^{\{2\}}|\alpha'_i))$ on $\mathfrak{g}_{\alpha'_i}$ for all new simple roots α'_i, $i \in I$, it will induce the correct charges for all new generators. Here, we should choose $\Lambda'^{\{2\}}$ so that it has scalar product 0 with α'_8, 1 with α'_6 and $n-2$ with α'_7, which suggests to take:

$$\Lambda'^{\{2\}} = \Lambda'^6 + (n-2)\Lambda'^7 \,. \tag{9.2}$$

Note first that the same set of charges can be obtained with all choices of the form: $\Lambda'^{\{2\}} = \ (c_1 n + 1)\Lambda'^6 + (c_2 n - 2)\Lambda'^7 + c_3 n \Lambda'^8$ for any set of $\{c_i\}_{i=1}^3 \in \mathbb{Z}^3$. In particular, there exists one weight vector that is valid for automorphisms of all finite orders, here $\Lambda'^{\{2\}} = \Lambda'^6 - 2\Lambda'^7$. However, in equation (9.2), we took all coefficients in $\mathbb{Z}_n$ as is required for the Kac-Peterson method to work. Since the $\mathfrak{a}_1$ is obviously invariant, we can restrict our attention to the $\mathfrak{a}_2$ part. One can verify that $\Lambda'^{\{2\}}$ satisfies the condition (9.1) since $\theta_{A_2} = \alpha'_6 + \alpha'_7$ implies $(\Lambda'^{\{2\}}|\theta_{A_2}) = n-1$.

In general, for a U-duality group G, we can define an $(r+1)$-th component of Λ' as $l_9^G = n - (\Lambda', \theta_G)$ ($l_9^{A_1 \times A_2} = 1$ in the above case). On the basis of this extended vector, one can apply the following diagrammatic method to obtain the invariant subalgebra in the finite-dimensional case (a simple justification of this method can be found in [29]):

- Replace the Dynkin diagram of $\mathfrak{g}$ by its extended Dynkin diagram, adding an extra node corresponding to the (non-linearly independent) root $\alpha'_9 = -\theta_G$. We denote the extended diagram by $\mathfrak{g}^+$ to distinguish it from the affine $\hat{\mathfrak{g}}$ in which the extra node $\alpha_0 = \delta - \theta_G$ is linearly independent.

- Discard all nodes corresponding to roots α'_i such that $l_i \neq 0$ and keep all those such that $l_i = 0$, $i \in \{9-r; \ldots; 9\}$.

- Let p be the number of discarded nodes, the (usually reductive) subalgebra left invariant by the automorphism $\mathcal{U}_2^{\mathbb{Z}_n}$ is given by the (possibly disconnected) remaining diagram times $p-1$ abelian subalgebras.

In particular, for $T^2/\mathbb{Z}_n$ in T^3 for $n \geq 3$, we see that l_6,l_7 and l_9 are non-zero, leaving invariant only $\alpha'_8 = \alpha_8$ which builds an $\mathfrak{a}_1$ diagram. Since $p = 3$, we should add two abelian factors for a total (complexified) invariant subalgebra $\mathfrak{a}_1 \oplus \mathbb{C}^{\oplus^2}$, the same conclusion we arrived at from Table 2. On the other hand, for $T^2/\mathbb{Z}_2$ in T^3, $l_7 = n-2 = 0$ and we have one more conserved node, leaving a total (complexified) invariant subalgebra $\mathfrak{a}_1^{\oplus^2} \oplus \mathbb{C}$.

Since the orbifold is not acting on other space-time directions, it seems logical to extend this construction by taking $\alpha'_i = \alpha_i$ and $l_i = 0$ $\forall i < 6$ for all $T^p \times T^2/\mathbb{Z}_n$ orbifolds. Indeed, for $p \leq 4$, we obtain $(\Lambda'^{\{2\}}|\theta_{A_4}) = n-1 \to l_9^{A_4} = 1$, $(\Lambda'^{\{2\}}|\theta_{D_5}) = n \to l_9^{D_5} = 0$ and $(\Lambda'^{\{2\}}|\theta_{E_6}) = n \to l_9^{E_6} = 0$, giving the results in Figure 2. Comparing Figure 2 with Table 3 in $D = 6$, we can identify $\alpha'_9 = -\theta_{D_5}$.

However, looking at the respective invariant subdiagrams in $D = 5$, it is clear that one should not choose $\alpha'_i = \alpha_i$ $\forall i < 6$. Looking at Table 3, one guesses that $\alpha'_3 = -\theta_{D_5}$, $\alpha'_4 = -\alpha_3$ and $\alpha'_5 = -\alpha_4$. Since there is only one element in the eigensubspace $\mathfrak{g}^{(n-2)}$, we also have to take $E'_7 = \frac{1}{2}(H_7 - i(E_7 + F_7))$, as before. On the other hand, there are now plenty of objects in $\mathfrak{g}^{(1)}$, all of them not commuting with E'_7. One should find one that commutes with $F_{\theta_{D_5}}$ and F_{α_3} but not with F_{α_4}. This suggests to set $E'_6 = \frac{1}{\sqrt{2}}(E_{34^25^268} - iE_{34^25^2678})$. Finally, we also take $E'_8 = F_5$. Computing the expression of the generator corresponding to the highest root in this new basis gives $E'_9 = F_{\theta_{E_6}} = iF_8$, as one would expect. Since the shift vector simply takes the same form $\Lambda'^{\{2\}} = \Lambda'^6 + (n-2)\Lambda'^7$ on a new basis of fundamental weights, the naive guess above was correct.

From $D = 4$ downwards, this ceases to be true, since naively $(\Lambda'^{\{2\}}|\theta_{E_7}) = 2n - 1 > n$. In fact, we should again change basis in $\mathfrak{e}_7$. Comparing again Figure 2 with Table 3, we see that there are 2 different equivalent ways to choose the 2 roots to be discarded, on the left (α'_2 and α'_3) or on the right (α'_7 and α'_9). We choose the latter, since it will be easier to generalize to $\mathfrak{e}_8$. Indeed, in the extended diagram of $\mathfrak{e}_8$, the Coxeter label of α'_9 will be the only one to be 1, making $l_9^{E_8} = n - 2$ the only possible choice. Further inspection of Figure 2 and Table 3 suggests to take the new basis as follows: $\alpha'_2 = -\alpha_8$, $\alpha'_3 = -\alpha_5$, $\alpha'_4 = -\alpha_4$, $\alpha'_5 = -\alpha_3$, $\alpha'_6 = -\theta_{D_5}$, $\alpha'_8 = -\alpha_2$ and finally

$$E'_7 = \frac{1}{\sqrt{2}}(E_{23^24^35^46^378^2} - iE_{23^24^35^46^37^28^2}) \in \mathfrak{g}^{(1)}\,.$$

A lengthy computation allows to show that this choice leads to
$E'_9 = F_{\theta'_{E_7}} = (i/2)(H_7 - i(E_7 + F_7)) \in \mathfrak{g}^{(n-2)}$ as it should, giving the shift vector: $\Lambda = \Lambda'^7$ with $l_9^{E_7} = n - 2$.

In $\mathfrak{e}_8$, a similar game leads to $\alpha'_2 = -\alpha_8$, $\alpha'_3 = -\alpha_5$, $\alpha'_4 = -\alpha_4$, $\alpha'_5 = -\alpha_3$, $\alpha'_6 = -\alpha_2$, $\alpha'_7 = -\alpha_1$, $\alpha'_8 = -\theta_{D_5}$ and finally

$$E'_1 = \frac{1}{\sqrt{2}}(E_{12^23^34^45^56^378^3} - iE_{12^23^34^45^56^37^28^3}) \in \mathfrak{g}^{(1)}\,,$$

leading to $E'_9 = F_{\theta'_{E_8}} = \frac{i}{2}(H_7 - i(E_7 + F_7)) \in \mathfrak{g}^{(n-2)}$ with shift vector $\Lambda' = \Lambda'^1$, while $l_9^{E_8} = n - 2$.

The results for $n = 2$ can be obtained directly by putting $l_7 = 0$, adding one more node to the diagrams instead of the abelian $\mathfrak{u}(1)$ factor. The results are summarized in Figure 3.

It was instructive to compare our method based on automorphisms induced by algebraic rotations and the standard classification of Lie algebra automorphisms based on shift vectors defining chief inner automorphisms. However, the mapping from one language to the other can be fairly obscure, in particular for orbifolds more complicated than the $T^2/\mathbb{Z}_n$ case treated above. In fact, in the $\mathfrak{a}_r$ serie of Lie algebras, for which the Coxeter labels are all equal, the necessary change of basis can be computed only once and trivially extended to larger algebras in the serie. In general, and in particular for exceptional algebras, one has to perform a different change of Cartan-Weyl basis whenever we consider the *same* orbifold in a larger U-duality symmetry algebra (or, geometrically speaking, when we compactify one more dimension).

Our method based on non-Cartan preserving automorphisms is thus more appropriate to treat a few particular orbifolds in a serie of algebras that are successively included

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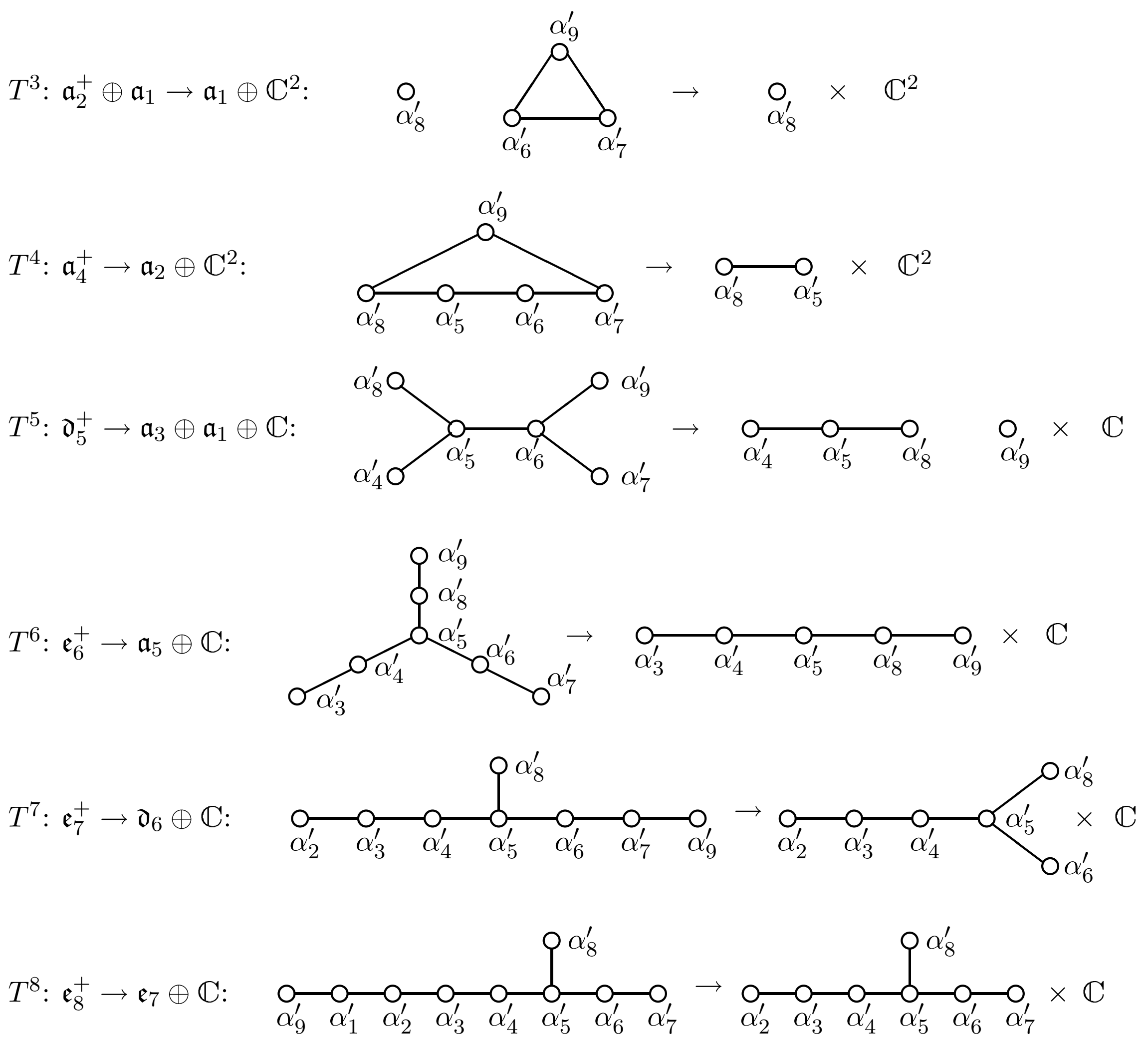

Figure 2: Diagrammatic method for $T^2/\mathbb{Z}_{n>2}$ orbifolds of M-theory

one into the other, as is the case for the U-duality algebras of compactified supergravity theories. On the other hand, the method based on chief inner automorphisms is more amenable to classify all orbifolds of a unique algebra, for example all possible breakings of a given gauge group under an orbifold action. For instance, the breakings of the $E_8 \times E_8$ gauge group of heterotic string theory have been treated this way by [27, 29]. It is also easier to generalize the method based on algebraic rotations to the infinite-dimensional case, since we can use the decomposition of $\mathfrak{e}_{10}$ in tensorial representations of $\mathfrak{sl}(10)$ and our intuition on the behaviour of tensorial indices under a physical rotation to identify non-trivial invariant objects.

We can draw a related conclusion from the explicit forms of the above basis transformation: when the orbifold is expressed in terms of the standard shift vector satisfying $(\Lambda|\theta) \leq n$, the geometric interpretation of the orbifold action gets blurred. More precisely, the directions in which the rotation is performed is determined above by the roots α'_i with coefficients $l_i = n - 2$. For example, in $\mathfrak{e}_7$, our original Lorentz rotation by $\mathcal{K}_{9\,10}$ repre-

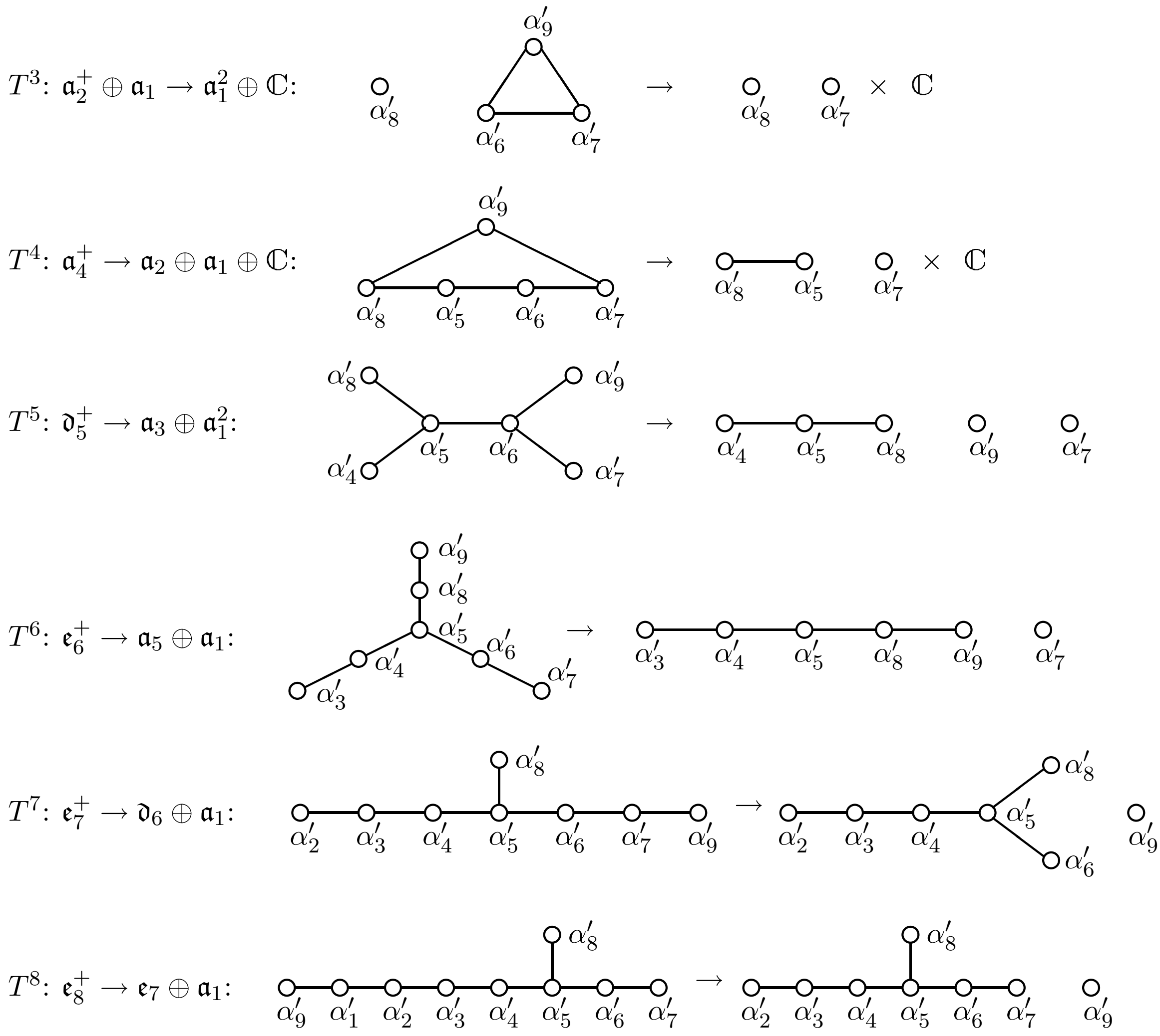

Figure 3: Diagrammatic method for $T^2/\mathbb{Z}_2$ orbifolds of M-theory

sented by α_7 appears in the standard basis as a gauge transformation generated by $\widetilde{\mathcal{Z}}_{456789}$. Similarly, in $\mathfrak{e}_8$, it seems that we are rotating in a direction corresponding to $(\widetilde{\mathcal{K}}_3)_{3456789\,10}$. Of course, mathematically, all conjugate Cartan-Weyl basis in a Lie algebra give rise to an isomorphic gradation of $\mathfrak{g}$, but the physical interpretation based on the decomposition of $\mathfrak{e}_r$ in tensorial representations of $\mathfrak{sl}(r)$ is obscured by the conjugation.

Indeed, our $T^q/\mathbb{Z}_2$ and $T^q/\mathbb{Z}_n$ orbifolds for $q = 2, 4$ all appear in the classification of $T^6/\mathbb{Z}_n$ orbifolds given in [27], where they are interpreted as $T^6/\mathbb{Z}_n$ orbifolds with particularly small breakings of the gauge group and degenerate shift vectors (in the sense of having lots of 0). It is however clear in our formalism that this degeneracy should actually be seen as having considered a rotation of null angle in certain directions.

9.2 Classes of shift vectors for $T^q/\mathbb{Z}_n$ orbifolds, for $q = 4, 6$

In the more complicated cases of $T^4/\mathbb{Z}_n$ and $T^6/\mathbb{Z}_n$ orbifolds, we will not give in detail the basis transformations necessary to obtain the standard shift vectors satisfying $(\Lambda'|\theta) \leq n$ for the whole serie of U-duality algebras. Rather, we will give the shift vectors in their universal form, which is valid for the whole serie of U-duality algebras. In particular, for $T^4/\mathbb{Z}_n$, the gradation of $\mathfrak{d}_5$ by eigensubspaces of $\mathcal{U}_4^{\mathbb{Z}_n}$ has been given in expressions (6.2), (6.3) and (6.4). A particularly natural choice of diagonal Cartan-Weyl basis for this decomposition is obtained by taking:

$$\begin{aligned}
E'_4 &= \tfrac{1}{\sqrt{2}}(E_4 + iE_{45})\,, & H'_4 &= \tfrac{1}{2}(2H_4 + H_5 - i(E_5 - F_5))\,,\\
E'_5 &= \tfrac{1}{2}(H_5 - i(E_5 + F_5))\,, & H'_5 &= i(E_5 - F_5)\,,\\
E'_6 &= \tfrac{1}{\sqrt{2}}(E_{56} - E_{67} + i(E_{567} + E_6))\,, & H'_6 &= \tfrac{1}{2}(H_5 + 2H_6 + H_7 - i(E_5 - F_5 + E_7 - F_7))\,,\\
E'_7 &= \tfrac{1}{\sqrt{2}}(H_7 - i(E_7 + F_7))\,, & H'_7 &= i(E_7 - F_7)\,,\\
E'_8 &= -\tfrac{1}{\sqrt{2}}(E_8 - iE_{58})\,, & H'_8 &= \tfrac{1}{2}(2H_8 + H_5 - i(E_5 - F_5))\,,
\end{aligned} \tag{9.3}$$

while F'_i is obtained from E'_i as above by conjugation and exchange of E and F. This leads to the following eigensubspace decomposition of $\mathfrak{d}_5$:

$$\begin{aligned}
\mathfrak{g}^{(0)} &= \mathrm{Span}\{H'_4; H'_5; H'_6; H'_7; H'_8; E'_6; E'_{567}; E'_{458}; E'_{4568}; E'_{45^2678}; E'_{45^26^278}; E' \leftrightarrow F'\}\,,\\
\mathfrak{g}^{(1)} &= \mathrm{Span}\{E'_4; E'_8; E'_{4567}; E'_{5678}; F'_{45}; F'_{58}; F'_{456}; F'_{568}\}\,,\\
\mathfrak{g}^{(2)} &= \mathrm{Span}\{E'_7; E'_{67}; E'_{45678}; F'_5; F'_{56}; F'_{45^268}\}\,.
\end{aligned} \tag{9.4}$$

The shift vector corresponding to this basis is given by:

$$\Lambda'^{\{4\}} = \Lambda'^4 + (n-2)\Lambda'^5 + 2\Lambda'^7 + \Lambda'^8\,,$$

which clearly reduces to $\Lambda'^{\{4\}} = \Lambda'^4 + \Lambda'^8$ in the case of $T^4/\mathbb{Z}_2$. By simply taking $E'_i = E_i$ for any additional roots that are unaffected by the orbifold action, this shift vector is valid in $\mathfrak{e}_r$, for $r = 6, .., 10$, as well.

For the case of $T^6/\mathbb{Z}_n$, we take the following Cartan-Weyl basis:

$$\begin{aligned}
E'_2 &= \tfrac{1}{\sqrt{2}}(E_2 + iE_{23})\,, & H'_2 &= \tfrac{1}{2}(2H_2 + H_3 - i(E_3 - F_3))\,,\\
E'_3 &= \tfrac{1}{2}(H_3 - i(E_3 + F_3))\,, & H'_3 &= i(E_3 - F_3)\,,\\
E'_4 &= \tfrac{1}{\sqrt{2}}(E_{34} + E_{45} - i(E_{345} - E_4))\,, & H'_4 &= \tfrac{1}{2}(H_3 + 2H_4 + H_5 + i(-E_3 + F_3 + E_5 - F_5))\,,\\
E'_5 &= \tfrac{1}{2}(H_5 + i(E_5 + F_5))\,, & H'_5 &= -i(E_5 - F_5)\,,\\
E'_6 &= \tfrac{1}{\sqrt{2}}(E_{56} - E_{67} - i(E_{567} + E_6))\,, & H'_6 &= \tfrac{1}{2}(H_5 + 2H_6 + H_7 + i(E_5 - F_5 + E_7 - F_7))\,,\\
E'_7 &= \tfrac{1}{\sqrt{2}}(H_7 + i(E_7 + F_7))\,, & H'_7 &= -i(E_7 - F_7)\,,\\
E'_8 &= -\tfrac{1}{\sqrt{2}}(E_8 + iE_{58})\,, & H'_8 &= \tfrac{1}{2}(2H_8 + H_5 + i(E_5 - F_5))\,,
\end{aligned} \tag{9.5}$$

that leads to the universal shift vector:

$$\Lambda'^{\{6\}} = \Lambda'^2 + (n-2)\Lambda'^3 + 2\Lambda'^5 + \Lambda'^6 + (n-4)\Lambda'^7 + (n-1)\Lambda'^8\,,$$

that is valid in $\mathfrak{e}_8$, $\mathfrak{e}_9$ and $\mathfrak{e}_{10}$, as well. It is obvious in this form that the degeneration of the coefficient l^7 when $n = 4$ leads to a larger invariant subalgebra with fewer abelian factors. On the other hand, as the invariant subalgebras for $T^6/\mathbb{Z}_4$ and $T^6/\mathbb{Z}_3$ both have no abelian factors, the coefficients of $\Lambda'^{\{6\}}$ does not allow to discriminate between them. Another fact worth noting is that setting $n = 2$ leads to $\Lambda'^{\{6\}} = \Lambda'^2 + \Lambda'^6 + \Lambda'^8$, corresponding to a $T^4/\mathbb{Z}_2$ orbifolds with respect to the nodes α_3 and α_5 and not to a $T^6/\mathbb{Z}_2$ orbifolds. This is natural since we chose the charge in the (x^9, x^{10})-plane to be $Q_3 = -2$, so that it reduces to the identity rotation for $n = 2$.

9.3 Roots of $\mathfrak{e}_{10}$ as physical class representatives

The universal shift vectors are mathematically interesting, but the original motivation to compute them was actually to give a physical interpretation of certain roots of $\mathfrak{e}_{10}$. Typically, our universal shift vectors Λ' are not roots, but we can use the self-duality of $Q(\mathfrak{e}_{10})$ and the periodicity modulo n of the orbifold action to replace Λ' by a root ξ generating the same orbifold action.

Self-duality of $Q(\mathfrak{e}_{10})$ relates the weight Λ' to a vector in the root lattice satisfying $(\Lambda'|\alpha') = (\tilde{\xi}|\alpha')\ \forall\alpha' \in \Delta(\mathfrak{g}, \mathfrak{h}')$. However, every root lattice vector is not a root. One should thus use the equivalence modulo n: $\Lambda' \equiv \Lambda' + n\sum_{i=-1}^{8} c_i\Lambda'^i = \overline{\Lambda}'$, for any 10-dimensional vector $\vec{c} \in \mathbb{Z}^{10}$, to find a weight $\overline{\Lambda}' = \sum_{i=-1}^{8} l_i\Lambda'^i$ such that:

$$\xi = \sum_{i,j=-1}^{8} (A^{-1})^{ij} l_j \alpha'_i$$

is a root of $\mathfrak{e}_{10}$. In fact, such a condition does not fix ξ uniquely either. However, it seems that there is a unique way to choose $\vec{c_q}$ so that $\xi^{[q,n]}$ is a root describing the orbifold $T^{10-q} \times T^q/\mathbb{Z}_n$ for all values of n.

From that point of view, we can see the shift vector as containing two parts: the *universal part*, that reflects the choices of orbifold directions and charges, and the *n-dependent part*, that defines the orbifold periodicity.

Concretely, it seems that $\vec{c_q}$ can always be chosen to be dual to a Weyl reflection of δ (at least for even orbifolds). In the case of $T^2/\mathbb{Z}_2$, for example, we had the universal part $\Lambda'^{\{2\}} = \Lambda'^6 - 2\Lambda'^7$, which is dual to $-\alpha'_7$. Adding $n(\Lambda'^7 - \Lambda'^8)$, i.e. the root $n\tilde{\delta}^{[2]} = n(\delta' + \sum_{i=-1}^{7} \alpha'_i)$, we obtain the desired form of shift vector in the physical basis:

$$\xi^{[2,n]} = (n, n, n, n, n, n, n, n, n-1, 1)\,.$$

From the tables of [52] it is easy to verify that this is a root of $\mathfrak{e}_{10}$ with $l = 3n$ for all values of $n \leq 6$, and it is very likely to be a root for any integer value of n. Note also that translating the results back in the original basis gives:

$$e^{i\frac{2\pi}{n}(\xi^{[2,n]}|\alpha')}\mathfrak{g}_{\alpha'} = e^{i2\pi(\alpha'|\tilde{\delta}^{[2]})}\,\mathrm{Ad}e^{\frac{2\pi}{n}(E_7-F_7)}\mathfrak{g}_{\alpha'}\,,$$

where the first factor does not contribute to the charge, so that the equivalence between the two descriptions, one in terms of shift vectors and the other in terms of non-Cartan preserving inner automorphisms, is obvious.

For $T^4/\mathbb{Z}_n$, we similarly take $\overline{\Lambda}'^{\{4\}} = \Lambda'^4 - 2\Lambda'^5 + 2\Lambda'^7 + \Lambda'^8 + n(\Lambda'^5 - \Lambda'^6 - \Lambda'^8)$, which corresponds to

$$\xi^{[4,n]} = -\alpha'_5 + \alpha'_7 + n(\delta' + \alpha'_{-1} + \alpha'_0 + \alpha'_1 + \alpha'_2 + \alpha'_3 + \alpha'_4 + \alpha'_5)$$

$$= (n, n, n, n, n, n, n-1, 1, n+1, n-1)\,.$$

Again, this is indeed a root $\forall n \leq 6$ and it can be checked to reduce to one of the 4 possible permutations proposed in [19] for $n = 2$. Furthermore, $H_{\xi^{[4,n]}} = nH_{\tilde{\delta}^{[4]}} - i(E_5 - F_5 - E_7 + F_7)$ as one would expect.

Finally, for $T^6/\mathbb{Z}_n$, one can check that $\overline{\Lambda}'^{\{6\}} = \Lambda'^2 - 2\Lambda'^3 + 2\Lambda'^5 + \Lambda'^6 - 4\Lambda'^7 - \Lambda'^8 + n(\Lambda'^7 - \Lambda'^8)$ has all desired properties. It is dual to

$$\begin{aligned}
\xi^{[6,n]} &= -\alpha'_3 + \alpha'_5 - 2\alpha'_7 + n(\delta' + \alpha'_{-1} + \alpha'_0 + \alpha'_1 + \alpha'_2 + \alpha'_3 + \alpha'_4 + \alpha'_5 + \alpha'_6 + \alpha'_7) \\
&= (n, n, n, n, n-1, n+1, n+1, n-1, n-2, 2)\,,
\end{aligned}$$

where the factor of -2 in front of α'_7 reminds us of the charge assignment $Q_3 = -2$. On the other hand, the - sign in front of α'_3 does not contradict our choice of $Q_1 = +1$, but is rather due to our Cartan-Weyl basis (9.5), in which H'_3 has a different conventional sign compared to H'_5 and H'_7. Accordingly, one obtains: $H_{\xi^{[6,n]}} = nH_{\tilde{\delta}^{[6]}} - i(E_3 - F_3 + E_5 - F_5 - 2(E_7 + F_7))$ as it should.

It is now easy to guess the general form of the shift vector for all $T^{10-q} \times T^q/\mathbb{Z}_n$ orbifolds, in which the orbifold projections are taken independently on each of the $(q/2)$ T^2 subtori (in other words, we exclude for example a $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold of T^6 for which one $\mathbb{Z}_3$ acts on the planes $\{x^5, x^6\}$ and $\{x^7, x^8\}$ and the other on the planes $\{x^7, x^8\}$ and $\{x^9, x^{10}\}$, since it contains two independent projections on the same T^2 subtorus).

By translating the tables of $\mathfrak{e}_{10}$ roots established by [52] in the physical basis, we can identify the roots which constitute class representatives of shift vectors (satisfying the conditions mentioned above) for orbifolds with various charge assignments, and build the classification represented in Tables 14 and 15. These listings deserve a few comments.

First of all, what we are really classifying are inner automorphisms of the type (4.2) with all different charges assignments (up to permutations of the shift vectors). Though some of these automorphisms allow to take a geometrical orbifold projection and descend to well-defined type IIA orbifolds, like the $T^4/\mathbb{Z}_n$ and the $T^6/\mathbb{Z}_n$ cases[14] we studied in Sections 6 and 7 for $n = 2, 3, 4, 6$, the Lefschetz fixed point formula would give a non-integer number of fixed points for most of the others. Clearly, such cases do not correspond to compactifications on geometrical orbifolds that can be made sense of in superstring theory (let alone preserve some supersymmetry). However, whether compactifications on such peculiar spaces makes sense in M-theory is, on the other hand, an open question. The invariant subalgebras and "untwisted" sectors can in any case be defined properly.

Second, we chose not to consider as different two shift vectors differing only by a permutation of $\tilde{\delta}$, but exhibiting the same universal part, for example $(3, 3, 3, 3, 3, 3, 3, 3, 2, 1)$ and $(3, 3, 3, 3, 3, 3, 3, 0, 2, 4)$.

Finally, looking at the Tables 14 and 15 in an horizontal way, one can identify series of shift vectors defining orbifold charges which appear as "subcharges" one of the others, when some Q_i's are set to zero. For example, starting from $T^8/\mathbb{Z}_6 \times T^2/\mathbb{Z}_3$ for $q = 10$,

[14] This kind of $T^6/\mathbb{Z}_n$ orbifold with charge assignment $(1, 1, -2)$ is denoted $T^4/\mathbb{Z}_n \times T^2/\mathbb{Z}_{n/2}$ in Tables 14 and 15 to distinguish it from the one with charge assignment $(1, -1, 1)$.

one obtains successively $T^6/\mathbb{Z}_6 \times T^2/\mathbb{Z}_3$, $T^4/\mathbb{Z}_6 \times T^2/\mathbb{Z}_3$, $T^2/\mathbb{Z}_6 \times T^2/\mathbb{Z}_3$ and $T^2/\mathbb{Z}_3$ for $q = 8, 6, 4, 2$, with shift vectors of monotonally decreasing squared lengths -8, -10, -12, -14 and -16.

Though, for evident typographical reasons, we were not able to accomodate all shift vectors related in this way on the same line, we have done so whenever possible to highlight the appearance of such families of class representatives. This explains the blank lines, whenever there was no such correspondence. An attentive study of Tables 14 and 15 shows that those families end up when a root of the serie reaches squared length 2. For example, going backwards and starting instead from $T^6/\mathbb{Z}_3'$ with a shift vector of null squared length for $q = 6$, one finds $T^2/\mathbb{Z}_6 \times T^6/\mathbb{Z}_3$ with a vector of squared length 2 for $q = 8$, but there is no $T^4/\mathbb{Z}_6 \times T^6/\mathbb{Z}_3$ for $q = 10$, since it would have to be generated by a vector of length 4, which is of course not a root.

To extend this classification to orbifolds that are not induced by an automorphism of type (4.2), further computations are nevertheless necessary (to obtain the correct form of the universal parts). However, exactly the same methods can in principle be applied and we leave this matter for further research. When tables of roots of $\mathfrak{e}_{10}$ will be available up to higher levels in α_8, one could also study orbifolds for higher values of n. Of physical interest are perhaps values of n up to 12, which would in principle require knowledge of roots of levels up to 36.

A more speculative question is whether these orbifold-generating roots all have another physical interpretation, for example as solitonic M-theory objects with or without non-trivial fluxes, just as in [19]. A first look at the general shape of these roots in the physical basis seems to confirm this view, since the first $(10 - q)$ n's remind of a $(10 - q)$-brane transverse to the orbifolded torus, while the other components might be given an interpretation as fluxes through the orbifold. Indeed, both are expected to contribute to local anomaly cancellation at the orbifold fixed points. We do not have a general realization of this idea, yet, but we will describe a number of more concrete constructions in the following and discuss in particular all of the $\mathbb{Z}_2$ cases in detail, hinting at a possible interpretation of the general $\mathbb{Z}_n$ ones.

10. $\mathbb{Z}_2$ orbifolds

The case of $\mathbb{Z}_2$-orbifolds is slightly degenerated and must be considered separately. In [21], the orbifold $T^{4m}/\mathbb{Z}_2$, $m = 1, 2$ and $T^{4m'+1}/\mathbb{Z}_2$, $m' = 0, 1, 2$ have been worked out, and the orbifold charges have been shown to be generated by a generic Minkowskian brane required [26, 25] for anomaly cancellation, living in the transverse space.

In this section, we will show how to treat all $T^q/\mathbb{Z}_2$ orbifolds for $q \in \{1, .., 9\}$. In Section 10.1, it will be shown how the algebraic results for invariant subalgebras in [19], which we henceforth refer to as $\mathbb{Z}_2$ orbifolds of the *first kind*, are recovered as a particular case in the more general framework of Section 9.

In Section 10.2, we investigate in detail the $q = 2, 3, 6, 7$ cases, or $\mathbb{Z}_2$ orbifolds of *the second kind*, which have not been considered in [19]. Let us stress that by *orbifolds of the second kind* we mean the purely algebraic implementation of the $\mathbb{Z}_2$ projection in the

U-duality algebra. Then, we will extract from the construction of Section 9.3 the roots of $\mathfrak{e}_{10}$ defining the representatives of classes of shift vectors for these orbifolds of M-theory and give a tentative physical interpretation.

Concretely, let i, j, k be transverse spacelike coordinates and A, B, C coordinates on the orbifold, under a $\mathbb{Z}_2$-transformation, $11D$-supergravity and fields have charge assignment

$$\begin{aligned} &(\text{all}) : g_{ij} \to +g_{ij}\,, \qquad g_{iA} \to -g_{iA}\,, \qquad g_{AB} \to +g_{AB}\,, \\ &\begin{pmatrix}\text{odd}\\ \text{even}\end{pmatrix} : C_{ijk} \to \mp C_{ijk}\,, \quad C_{ijA} \to \pm C_{ijA}\,, \quad C_{iAB} \to \mp C_{iAB}\,, \quad C_{ABC} \to \pm C_{ABC}\,, \end{aligned}$$

odd and *even* referring to the dimension of the orbifold torus.

In contrast to the $\mathbb{Z}_{n>2}$ case, where the inner automorphisms generating the orbifold charges were pure $SO(r)$ rotations, the action of a $\mathbb{Z}_2$-orbifold can be regarded as an element of the larger $O(r) = \mathbb{Z}_2 \times SO(r)$. This distinctive feature of $\mathbb{Z}_2$-orbifold can be ascribed to the fact that while even orbifolds act as central symmetries and may be viewed as π-rotations, hence falling in $O^+(r)$ (positive determinant elements connected to the identity), odd orbifolds behave as mirror symmetries, and thus fall in $O^-(r)$. Concretely, negative determinant orthogonal transformations will contain, in the $\mathfrak{e}_{10}$ language, a rotation in the α_8 direction, namely $\mathrm{Ad}(e^{\pi\,(E_8-F_8)})$ or $\mathrm{Ad}(e^{i\pi\,H_8})$, which, in this framework, behaves as a mirror symmetry.

The even case can be dealt with in a general fashion by applying the following theorem:

Theorem 10.1 *Let $T^q/\mathbb{Z}_2$ be a $\mathbb{Z}_2$ toroidal orbifold of $\mathfrak{e}_r$, for $q \in \{1, \dots, 9\}, r \in \{q+1, \dots, 10\}$. Let q be either $2m$ or $2m+1$. Given a set of (possibly non-simple) roots $\Delta_{\mathbb{Z}_2} = \{\beta_{(p)}\}_{p=1,..,m}$ satisfying $(\beta_{(p)}|\beta_{(l)}) = c_p\delta_{p,l}$, with $c_p \leqslant 2$ and provided the orbifold acts on the U-duality algebra $\mathfrak{g}^U$ with the operator $\mathcal{U}^{\mathbb{Z}_2}_{2m} \in G^U$ defined, according to expression (4.2), by*

$$\mathcal{U}^{\mathbb{Z}_2}_q = \prod_{p=1}^{m} \mathrm{Ad}(e^{\pi\,(E_{\beta_{(p)}} - F_{\beta_{(p)}})})\,, \tag{10.1}$$

then, the orbifold action decomposes on the root-subspace $\mathfrak{g}^U_\alpha \subset \mathfrak{g}^U$ as

$$\mathcal{U}^{\mathbb{Z}_2}_q \cdot \mathfrak{g}^U_\alpha \equiv \prod_{p=1}^{m} \mathrm{Ad}(e^{i\pi\,H_{\beta_{(p)}}}) \cdot \mathfrak{g}^U_\alpha = (-1)^{\sum_{p=1}^m (\beta_{(p)}|\alpha)}\,\mathfrak{g}^U_\alpha\,, \qquad \forall \alpha \in \Delta(\mathfrak{g}^U)\,. \tag{10.2}$$

If the $\mathbb{Z}_2$-orbifold is restricted to extend in successive directions, starting from x^{10} downwards, it can be shown that for any root $\alpha = \sum_{j=-1}^{8} k^j \alpha_j \in \Delta(\mathfrak{g}^U)$, expression (10.2) assumes the simple form

$$\mathcal{U}^{\mathbb{Z}_2}_{2m} \cdot \mathfrak{g}^U_\alpha \equiv \mathrm{Ad}(e^{i\pi\,\sum_{i=1}^m H_{9-2i}}) \cdot \mathfrak{g}^U_\alpha = \begin{cases} (-1)^{k^6}\,\mathfrak{g}^U_\alpha\,, & \text{for } m = 1 \\ (-1)^{k^{8-2m}+k^8}\,\mathfrak{g}^U_\alpha\,, & \text{for } 2 \leqslant m \leqslant 5 \end{cases}. \tag{10.3}$$

for even orbifold. For odd ones, we note the appearance of the mirror operator we mentioned above

$$\mathcal{U}^{\mathbb{Z}_2}_{2m+1} \cdot \mathfrak{g}^U_\alpha \equiv \mathrm{Ad}\left(e^{i\pi\,(H_8 + H_6 + \sum_{i=1}^m H_{8-2i})}\right) \cdot \mathfrak{g}^U_\alpha = (-1)^{k^{7-2m}}\,\mathfrak{g}^U_\alpha\,, \ \text{for } m \geqslant 0\,. \tag{10.4}$$

Following Section 9.1, we are free to recast the orbifold charges resulting from expressions (10.3) and (10.4), by resorting to a shift vector $\xi^{[q,2]}$ such that:

$$\mathcal{U}_q^{\mathbb{Z}_2} \cdot \mathfrak{g}_\alpha^U = (-1)^{(\xi^{[q,2]}|\alpha)}\, \mathfrak{g}_\alpha^U\,, \;\; \forall q = 1, \ldots, 9\ .$$

The subalgebra invariant under $T^q/\mathbb{Z}_2$ is now reformulated as a KMA with root system

$$\Delta_{\text{inv}} = \Big\{ \alpha \in \Delta(\mathfrak{g}^U)\,|(\xi^{[q,2]}|\alpha) = 0 \bmod 2 \Big\}\ . \tag{10.5}$$

This is definition of the $\mathbb{Z}_2$-charge used in [21, 19].

10.1 $\mathbb{Z}_2$ orbifolds of M-theory of the first kind

The orbifolds of M-theory with $q = 1, 4, 5, 8, 9$ have already been studied in [19], and a possible choice of shift vectors has been shown to be, in these cases, dual to prime isotropic roots, identified in [21] as Minkowskian branes. As such, they were interpreted as representatives of the 16 transverse M-branes stacked at the 2^q orbifold fixed points and required for anomaly cancellation in the corresponding M-theory orbifolds [25, 26].

In this section about $\mathbb{Z}_2$ orbifolds of the first kind, we will show how to rederive the results of [19] about shift vectors and invariant subalgebra, from the more general perspective we have developed in Section 9 by resorting to the Kac-Peterson formalism. After this cross check, we will generalize this construction to the $q = 2, 3, 6, 7$ cases, which have not been considered so far, and show how the roots $\xi_2^{[q,2]}$ are related to D-branes and involved in the cancellation of tadpoles due to O-planes of certain type 0B' orientifolds. For this purpose, we start by summarizing in Table 16 the shift vectors for the $q = 1, 4, 5, 8, 9$ cases found in [19], specifying in addition the $SL(10,\mathbb{R})$-representation they belong to. Next we will show how the results of Table 16 for even orbifolds can be retrieved as special cases of the general solutions computed in Section 9.1.

The root of E_{10} relevant to the $q = 4$ orbifold can be determined as a special case of $T^4/\mathbb{Z}_n$ shift vectors, namely:

$$\xi^{[4,2]} = 2(\Lambda^5 - \Lambda^6 - \Lambda^8) + \Lambda^{\{4\}} = 2\tilde{\delta}^{[4]} - \alpha_5 + \alpha_7\,,$$

which coincides with the results of Table 16. This choice of weight is far from unique, but is the lowest height one corresponding to a root of E_{10} (given that $\Lambda^{\{4\}}$ is not a root). Likewise $\xi^{[8,2]}$ can in principle be deduced from the generic weight $\Lambda^{\{8\}}$ determining the $T^8/\mathbb{Z}_n$ charges.

Shift vectors for odd orbifolds of Table 16 can also be recast in a similar form, even though they do not generalize to $n > 2$. We can indeed rewrite:

$$\xi^{[1,2]} = 2(2\Lambda^7 - \Lambda^6) - \Lambda^7, \qquad \xi^{[5,2]} = 2(-\Lambda^8) + \Lambda^3, \qquad \xi^{[9,2]} = 2(-\Lambda^{-1}) + \Lambda^{-1}.$$

The last four E_{10} roots listed in Table 16 were identified in [21] as, respectively, Minkowskian Kaluza-Klein monopole (KK7M), M5-brane, M2-brane and Kaluza-Klein particle (KKp), with spatial extension in the transverse directions and have been presented in Table 1. The first root is the mysterious M-theory lift of the type IIA D8-brane, denoted as KK9M in this paper. In the language of Table 5, these roots correspond to the representation weights $(A \otimes \widetilde{K})_{(99)[1\cdots 9]}$, $D_{(10)[1\cdots 6\,10]}$, $B_{(1)[2\cdots 5]}$, $A_{(12)}$, $\widetilde{K}_{(2)[3\cdots 10]}$.

Furthermore, it has been shown in [25, 26] that the consistency of $\mathbb{Z}_2$ orbifolds of M-theory of the first kind requires the presence at the fixed points of appropriate solitonic configurations. For $T^{q=5,8}/\mathbb{Z}_2$, one needs respectively 16 M5-branes/M2-branes to ensure anomaly cancellation. In the case of $T^9/\mathbb{Z}_2$, 16 units of Kaluza-Klein momentum are needed, while Kaluza-Klein monopoles with a total Chern class of the KK gauge bundle amounting to 16 is required in the case of $T^4/\mathbb{Z}_2$.

For $q = 4, 5, 8, 9$, the transverse Minkowskian objects of Table 16 having all required properties were interpreted as generic representatives of these non-perturbative objects. However, their total multiplicity/charge cannot be inferred from the shift vectors. It was proposed in [19] that these numbers could be deduced from an algebraic point of view from the embedding of $\mathfrak{g}_{\rm inv}$ into a real form of the conjectured heterotic U-duality symmetry $\mathfrak{de}_{18}$. However, this idea seems to be difficult to generalize to the new examples treated in the present paper and we will not discuss it further.

For $q = 1$, the analysis is a bit more subtle, and needs to be carried out in type IIA language. To understand the significance of the shift vector in this case, it is convenient to reduce from M-theory on $T^8 \times S^1/\mathbb{Z}_2 \times S^1$ to type IIA theory on $T^8 \times S^1/\Omega I_1$, where I_1 is the space parity-operator acting on the S^1 as the original $\mathbb{Z}_2$ inversion, while Ω is the world-sheet parity operator. In this setup, the appropriate shift vector is $\xi_\sigma^{[1,2]} = (2,2,2,2,2,2,2,2,2,1,4)$, which can be interpreted as a KK9M of M-theory with mass:

$$M_{\rm KK9M} = M_p^{-9} V^{-1} e^{\langle \xi | H_m \rangle} = M_p^{12} R_1 \cdots R_8 R_{10}^3 \,.$$

Upon reduction to type IIA theory, we reexpress it in string units by setting

$$R_{10} = g_A M_s^{-1}\,, \quad \text{and} \quad M_p = g_A^{-1/3} M_s\,, \tag{10.6}$$

and take the limit $M_p R_{10} \to 0$:

$$M_{\rm KK9M} \to M_{D_8} = \frac{M_s^9}{g_A} R_1 \cdots R_8 \,.$$

The resulting mass is that of a D8-brane of type IIA theory. The appearance of this object reflects the need to align 8 D8-branes on each of the two O8 planes at both ends of the orbifold interval to cancel locally the -8 units of D8-brane charge carried by each O8 planes, a setup known as type I' theory.

The chain of invariant subalgebras $\mathfrak{g}_{\rm inv}$ in Table 17 is obtained by keeping only those root spaces of $\mathfrak{g}^U$ which have eigenvalue $+1$ under the action of $\mathcal{U}_{2m}^{\mathbb{Z}_2}$ (10.3) or $\mathcal{U}_{2m+1}^{\mathbb{Z}_2}$ (10.4). We can use the set of invariant roots (10.5) to build the Dynkin diagram of $\mathfrak{g}_{\rm inv}$, but this is not enough to determine root multiplicities in $D = 1$. In the hyperbolic case indeed, one will need to know the dimension of the root spaces $\mathfrak{g}_\alpha^U \subset \mathfrak{g}^U = \mathfrak{e}_{10|10}$ which are invariant under the actions (10.3) or (10.4) to determine the size of $\mathfrak{g}_{\rm inv}$. We will come back to this issue at the end of this section.

This construction of the root system leads, $\forall q = 1,4,5,8,9$, to a unique chain of invariant subalgebras, depicted in Table 17. Thus, we verify that the statement made in [19] for the hyperbolic case is true for all compactifications of type $T^{11-(D+q)} \times T^q/\mathbb{Z}_2$ with $q = 1,4,5,8,9$. There, this isomorphism was ascribed to the fact that, in $D = 1$, the shift vectors (16) are all prime isotropic and thus Weyl-equivalent to one another. The mathematical origin of this fact lies in the general method developed by Kac-Peterson explained in Section 9.1, which states that equivalence classes of shift vectors related by Weyl transformation and/or translation by n times any weight lattice vector lead to isomorphic fixed-point subalgebras. Since a Weyl reflection in $\mathfrak{g}^U$ generates a U-duality transformation in M-theory and its low-energy supergravity, this isomorphism seems to indicate that all such orbifolds are dual in M-theory, as pointed out in [26, 76]. In fact, if one had chosen to reduce $\mathbb{Z}_2$ orbifolds of the first kind on a toroidal direction for q odd, and on an orbifolded direction for q even, one would have realized that they are all part of the serie of mutually T-dual orientifolds (a T-duality on x^i is denoted by $\mathcal{T}_i$):

$$\begin{aligned}&\text{type IIB on } T^9/\Omega \xrightarrow{\mathcal{T}_9} \text{type IIA on } T^9/\Omega I_1 \xrightarrow{\mathcal{T}_8} \text{ type IIB on } T^9/(-1)^{F_L}\Omega I_2 \xrightarrow{\mathcal{T}_7} \\ &\quad \to \text{ type IIA on } T^9/(-1)^{F_L}\Omega I_3 \xrightarrow{\mathcal{T}_6} \text{ type IIB on } T^9/\Omega I_4 \xrightarrow{\mathcal{T}_8} \dots\end{aligned} \tag{10.7}$$

where I_r denotes the inversion of the last r space-time coordinates, while Ω is as usual the world-sheet parity. The *space-time* left-moving fermions number $(-1)^{F_L}$ appear modulo 4 in these dualities.

The reality properties of $\mathfrak{g}_{\text{inv}}$ are easy to determine. Since the original balance between Weyl and Borel generators is preserved by the orbifold projection, the non-abelian part of $\mathfrak{g}_{\text{inv}}$ remains split. In $D = 8,\dots,5$, the abelian $\mathfrak{so}(1,1)$ factor in $\mathfrak{g}_{\text{inv}}$ is generated by the non-compact element $H^{[q]}$ which also appears in $T^q/\mathbb{Z}_{n>2}$ orbifolds. For $q = 4$, it is, for instance, given by $H^{[4]} = H_8 - H_4$ in $D = 6$ and $2H_3 + 4H_4 + 3H_5 + 2H_6 + H_7$ in $D = 5$, as detailed in Section 6, and is enhanced, in $D = 4$, to the $\mathfrak{sl}(2,\mathbb{R})$ factor appearing in Table 17 when $H^{[4]}$ becomes the root $\gamma = \alpha_{23^24^35^36^278} \in \Delta_+(\mathfrak{e}_7)$. The procedure is similar for $q = 1,5,8,9$, for different combinations $H^{[q]}$ and positive roots γ.

The root multiplicities in $\mathfrak{g}_{\text{inv}}$ is only relevant to the two cases $D = 2,1$, for which the root multiplicities are inherited from $\mathfrak{e}_9$ and $\mathfrak{e}_{10}$. For $\mathfrak{g}^U = \mathfrak{e}_9$ we have $\mathfrak{g}_{\text{inv}} = \hat{\mathfrak{d}}_8$, since $\delta_{\mathfrak{g}_{\text{inv}}} = \delta$, and since δ and $\delta_{\hat{\mathfrak{d}}_8}$ both have multiplicity 8.

In $D = 1$, the story is different. In [19], it has been shown that $\mathfrak{g}_{\text{inv}}$ contains a subalgebra of type $\mathfrak{de}_{10}$. The authors have performed a low-level decomposition of both $\mathfrak{g}_{\text{inv}}$ and $\mathfrak{de}_{10}$. For a generic over-extended algebra $\mathfrak{g}^{\wedge\wedge}$, such a decomposition with respect to its null root $\delta_{G^{\wedge\wedge}}$ is given by $(\mathfrak{g}^{\wedge\wedge})_{[k]} \doteq \bigoplus_{(\alpha,\delta_{G^{\wedge\wedge}})=k} \mathfrak{g}^{\wedge\wedge}_\alpha$. In particular, they define: $(\mathfrak{g}_{\text{inv}})_{[k]} \doteq \mathfrak{g}_{\text{inv}} \cap (\mathfrak{e}_{10})_{[k]}$ and show that:

$$(\mathfrak{g}_{\text{inv}})_{[1]} \simeq (\mathfrak{de}_{10})_{[1]}\,, \qquad (\mathfrak{g}_{\text{inv}})_{[2]} \supset (\mathfrak{de}_{10})_{[2]}\,.$$

The first equality is a reformulation of $\mathfrak{g}_{\text{inv}} = \hat{\mathfrak{d}}_8$ for $\mathfrak{g}^U = \mathfrak{e}_9$. The second result comes from the fact that the orbifold projection selects certain preserved root subspaces without affecting their dimension. This feature is similar to what we have observed in the case of $\mathbb{Z}_n$ orbifolds, where the original root multiplicities are restored after modding out the Borcherds or KM algebras appearing in $D = 1$ by their centres and derivations.

10.2 $\mathbb{Z}_2$ orbifolds of M-theory of the second kind and orientifolds with magnetic fluxes

Let us first recall that by *orbifolds of the second kind* we mean the purely algebraic implementation of T^{10}/I_q, $q = 2, 3, 6, 7$, in the U-duality algebra. In this case, the connection to orbifolds of M-theory will be shown to be more subtle than in Section 10.1. Indeed, since the algebraic orbifolding procedure does not discriminate between two theories with the same bosonic untwisted sectors and different fermionic degrees of freedom, there are in principle several candidate orbifolds on the M-theory side to which these orbifolds of the second kind could be related.

The first (naive) candidate one can consider is to take M-theory directly on T^{10}/I_q, $q = 2, 3, 6, 7$. Then following the analysis of Section 10.1, a reduction of such orbifolds to type II string theory would result in a chain of dualities similar to expression (10.7), with the important difference that the $(-1)^{F_L}$ operator now appears in the opposite places. It is well known that the spectrum of such theories cannot be supersymmetric. Referring to the chain of dualities (10.7) with the required extra factor of the left-moving fermionic number operator, one observes that such theories do not come from a consistent truncation of type IIB string theory, since $(-1)^{F_L}\Omega$ is not a symmetry thereof, and all of them are therefore unstable.

A more promising candidate is M-theory on $T^q/\{(-1)^F S, \mathbb{Z}_2\}$, where $(-1)^F$ is now the total *space-time* fermion number and S represents a π shift in the M-theory direction. In contrast to the preceding case, these orbifolds are expected to be dual to orientifolds of type 0 theory, which are non-supersymmetric but are nonetheless believed to be stable, so that tadpole cancellation makes sense in such setups.

Let us work out, in these type 0 cases, a chain of dualities similar to expression (10.7). To start with, we review the argument stating that M-theory on $S^1/(-1)^F S$ is equivalent to the non-supersymmetric type 0A string theory in the small radius limit [77].

Considering the reduction of M-theory on $S^1 \times S^1/(-1)^F S$ to type IIA string theory on $S^1/(-1)^F S$, one can determine the twisted sector of this orbifold (with no fixed point) and perform a flip on $\{x^9, x^{10}\}$ to obtain the spectrum of M-theory on $S^1/(-1)^F S$. At the level of massless string states, all fermions are projected out from the untwisted sector and there appears a twisted sector that doubles the RR sector and adds a NSNS tachyon, leading to type 0A string theory in 10 dimensions. Interestingly, type 0 string theories have more types of $\mathbb{Z}_2$ symmetries and thus more consistent truncations. In $D = 10$, type 0A theory is symmetric under the action of Ω, while type 0B theory is symmetric under Ω, $\Omega(-1)^{f_L}$ and $\Omega(-1)^{F_L}$, where f_L and F_L are respectively the *world-sheet* and *space-time* left-moving fermion numbers. Furthermore, their compactified versions on T^{10} each belong to a serie of orientifold theories similar to (10.7). Among these four chains of theories, one turns out to be tachyon-free, the chain descending from type 0B string theory on $\Omega(-1)^{f_L}$. Let us concentrate on this family of orientifolds and show that the M-theory orbifolds of the second kind can all be seen to reduce to an orientifold from this serie in the small radius limit.

In order to see this, one can mimick the procedure used for (10.7) and reduce on a toroidal direction for q odd, and on an orbifolded direction for q even. The untwisted sectors of our orbifolds then turns out to correspond to those of an orientifold projection by $\Omega(-1)^{f_L} I_{(q)}$, resp. $\Omega(-1)^{f_R} I_{(q-1)}$, on type 0A string theory. A projection by $\Omega(-1)^{f_{L/R}} I_{(q)}$ has the following effects in type 0A string theory: it eliminates the tachyon and half of the doubled RR sector, the remaining half being distributed over the untwisted and twisted sectors of M-theory on $S^1/(-1)^F S$. Consequently, one expects to obtain theories that belong to the following chain of dual non-supersymmetric orientifolds:

$$\begin{aligned}&\text{type 0B on } T^9/(-1)^{f_L}\Omega \xrightarrow{\mathcal{T}_9} \text{type 0A on } T^9/(-1)^{f_R}\Omega I_1 \xrightarrow{\mathcal{T}_8} \text{type 0B on } T^9/(-1)^{f_R}\Omega I_2 \\ &\xrightarrow{\mathcal{T}_7} \text{ type 0A on } T^9/(-1)^{f_L}\Omega I_3 \xrightarrow{\mathcal{T}_6} \text{ type 0B on } T^9/(-1)^{f_L}\Omega I_4 \xrightarrow{\mathcal{T}_8} \dots\end{aligned} \tag{10.8}$$

where $(-1)^{f_L}$ and $(-1)^{f_R}$ again appear modulo 4 in these dualities.[15] Complications might however arise at the twisted sector level when reducing to type 0A string theory on an orbifolded direction, since one should take into account a possible non-commutativity between the small radius limit and the orbifold limit. We will come back to this point later.

Instead, we first want to remind the reader that, as was shown in [34], type 0B string on $(-1)^{f_L}\Omega$ can be made into a consistent non-supersymmetric string theory by cancelling the tadpoles from the two RR 10-forms through the addition of 32 pairs of D9- and D9'-branes for a total $U(32)$ gauge symmetry. This setup is usually called type 0'. There is also a NSNS dilaton tadpole, but this does not necessarily render the theory inconsistent. Rather, it leads to a non-trivial cosmological constant through the Fischler-Susskind mechanism [78, 79]. It was also shown in [34] that there is no force between the D9- and D9'-branes and that twisted sector open strings stretched between them lead to twisted massless fermions in the $\mathbf{496} \oplus \overline{\mathbf{496}}$ representation of $U(32)$. Even though the latter are chiral Majorana-Weyl fermions, it was shown in [80, 81] that a generalized Green-Schwarz mechanism ensures anomaly cancellation.

To characterize the twisted sectors of such orientifolds of type 0' string theory algebraicly, we will again use the equivalence classes of shift vectors that generate the orbifolds on the U-duality group. The simplest elements of these classes which are also roots give the set of real roots in Table 18.

As is obvious from the second and third column, all such roots are in $\Delta_+(\mathfrak{e}_8)$ and correspond to instantons completely wrapping the orbifolded torus. Since they are purely $\mathfrak{e}_8$ roots, we do not expect them to convey information on the string theory twisted sectors. As such, this set of shift vectors does not lend itself to an interesting physical interpretation, but gives however certain algebraic informations. All these roots being in the same orbit of the Weyl subgroup of E_8, the resulting invariant subalgebras are again isomorphic (when existing) $\forall q = 2, 3, 6, 7$. We list the invariant subalgebras for M-theory on $T^2/\{(-1)^F S, \mathbb{Z}_2\}$

[15] From the point of view of M-theory, these dualities sometimes exchange the untwisted and twisted sectors under $(-1)^F S$.

together with their Dynkin diagrams in Table 19. The same invariant subalgebras appear for all values of q, but of course start to make sense only in lower dimensions.

The invariant subalgebras are not simple for $D \geq 3$ and all of them contain at least one $\mathfrak{sl}(2,\mathbb{R})$ factor with simple root $\tilde{\xi}^{[q,2]}$. When an abelian factor is present, it coincides with the non-compact Cartan element $H^{[q]}$ encountered in $T^q/\mathbb{Z}_{n>2}$ orbifolds. Furthermore, in contrast to the connected $\hat{\mathfrak{d}}_8$ diagram obtained for $q = 1,4,5,8,9$, the invariant subalgebra for $q = 2,3,6,7$ is given in $D = 2$ by an affine central product, as in the $T^{2,4,6}/\mathbb{Z}_{n>2}$ cases treated before. In $D = 1$, the invariant subalgebra is the following quotient of the KMA whose Dynkin diagram is drawn in Table 19:

$$\mathfrak{g}_{\text{inv}} = {}^2\mathcal{KM}_{11|12}/\{\mathfrak{z}, d_1\}\,,$$

where $\mathfrak{z} = c_{\mathfrak{e}_7} - c_{\hat{\mathfrak{a}}_1}$. As in the $T^6/\mathbb{Z}_{3,4}$ cases, ${}^2\mathcal{KM}_{11}$ has a singular Cartan matrix with similar properties.

We will now show that certain equivalent shift vectors can be interpreted as configurations of D9 and D9'-branes cancelling R-R tadpoles in a type 0' string theory orientifold. This can be achieved by adding an appropriate weight lattice vector $\Lambda^{[q]}$ to $\tilde{\xi}^{[q,2]}$ that do not change the scalar products modulo 2. It should be chosen so that $\Lambda^{[q]} + \tilde{\xi}^{[q,2]}$ is a root, and gives insight on the possible M-theory lift of such constructions. More precisely, we want to convince the reader that certain choices of shift vectors generating M-theory orbifolds of the second kind can be seen as representing either magnetized D9-branes or their image D9'-branes in some type 0' theory with orientifold planes. Such branes carry fluxes that contribute to the overall D$(9-q)$-brane charge for even q and D$(10-q)$-brane charge for odd q, but not to the higher ones.

Let us first study the example of a $T^3/\mathbb{Z}_2 \times S^1/(-1)^F S$ orbifold of M-theory. Following the above construction, it should reduce in the limit $M_P R_{10} \to 0$ to type 0A string theory on $T^6 \times T^3/(-1)^{f_L}\Omega I_3$, which is T-dual to type 0B string theory on $T^7 \times T^2/(-1)^{f_R}\Omega I_2$. We summarize these dualities in the diagram below:

$$\begin{array}{l}
\text{M-theory on } T^6 \times T^3/\mathbb{Z}_2 \times S^1/(-1)^F S \\
\qquad\qquad\qquad \Big\downarrow {\scriptstyle M_P R_{10} \to 0} \\
\text{type 0A on } T^6 \times (S^1 \times T^2)/(-1)^{f_L}\Omega I_3 \xrightarrow{\ \mathcal{T}_7\ } \text{type 0B on } T^6 \times S^1 \times T^2/(-1)^{f_R}\Omega I_2
\end{array}$$

In this last type 0B orientifold, there will be one orientifold plane carrying -4 units of D7- and D7'-brane charge at each of the four orbifold fixed points. Suppose that we consider N pairs of magnetized D9- and D9'-branes carrying fluxes in the orbifolded plane (x'^8, x^9). This system induces two Chern-Simons couplings on the world-volume of the space-time filling branes:

$$\frac{M_s^{10}}{2(2\pi)^9}\int_{\mathbb{R}\times T^9} C_8 \wedge 2\pi\alpha' \text{Tr}(F_2) = \frac{M_s^8}{(2\pi)^7}\int_{\mathbb{R}\times T^7} C_8 \cdot \frac{1}{2\pi}\int_{T^2/\mathbb{Z}_2} \text{Tr}(F_2)\,,$$

and a similar expression involving C'_8. The quantized fluxes can then be chosen in such a way that the resulting total positive D7- and D7'-brane charges cancel the negative charges from the orientifold planes and ensure tadpole cancellation. Note that these charges are determined by the first Chern class c_1 of the $U(N)$ gauge bundle.

We can use an analogy with the supersymmetric case, where the system of O7-planes and magnetized D9-branes in a $T^8 \times T^2/(-1)^{F_L}\Omega\mathbb{Z}_2$ type IIB orientifold has a well-known T-dual equivalent [36, 82] built from D8-branes at angle with O8-planes in a $T^9 \times S^1/(-1)^{F_L}\Omega\mathbb{Z}_2$ type IIA orientifold, in which the flux is replaced by an angle χ in the following way:

$$2\pi\alpha' F_{89} = \frac{c_1}{N}\frac{\mathbb{1}_N}{M_s^2 R_8' R_9} \xrightarrow{\mathcal{T}_8} \cot(\chi) = \frac{c_1}{N}\frac{R_8}{R_9} \tag{10.9}$$

and where the type IIB radius[16] is $R_8' = 1/M_s^2 R_8$. In fact, the appearance of D7-brane charges in the absence of D9-brane ones on the type IIB side can be understood, in the dual setup, as the addition (resp. cancellation) of the charges due to the tilted D8-branes to those of their image branes with respect to the orientifold O8-plane. The image D8-brane indeed exhibits a different angle, being characterized by wrapping numbers $(c_1, -N)$ instead of (c_1, N) around the directions along the orbifold. In our non-supersymmetric case, however, one should keep in mind that the image brane of a D9-brane under $\Omega(-1)^{f_R}$ is a D9'-brane.

We will now show that the appropriate shift vector encodes not only explicit information on the presence of type 0' pairs of D9- and D9'-branes, but also on the tilting of their dual type 0A D8- and D8'-branes with respect to the O8-planes. One can then deduce the presence of fluxes from the angle χ.

To understand how this comes about, we note that both string theory D9- and D8-branes descend from the (conjectured) KK9M soliton of M-theory described by the $\mathfrak{e}_{10}$ roots that are permutations of $\xi = ((2)^6, 1, 2, 2, 4)$ in the following way:

$$M_{\mathrm{KK9M}} = M_p^{-9} V^{-1} e^{\langle \xi | H_R \rangle} = M_p^{12} R_1 \cdots R_6 R_8 R_9 R_{10}^3 \,.$$

Following the chain of dualities (10.14), we successively obtain the D8- and D9-brane mass formulae:

$$M_p^{12} R_1 \cdots R_6 R_8 R_9 R_{10}^3 \xrightarrow{M_p R_{10} \to 0} M_{\mathrm{D8}} = \frac{M_s^9}{g_A} R_1 \cdots R_6 R_8 R_9 \xrightarrow{\mathcal{T}_7} M_{\mathrm{D9}} = \frac{M_s^{10}}{g_B} R_1 \cdots R_6 R_7' R_8 R_9 \tag{10.10}$$

Now, we select one definite shift vector $\tilde{\xi}^{[3,2]}$ from all equivalent ones, which has the particularity to correspond like ξ to a root of level 4. It is obtained from a permutation of the root $\tilde{\xi}^{[3,2]}$ that describes an orbifold in the directions $(x^7; x^8; x^9)$, namely $\tilde{\xi}_\sigma^{[3,2]} = ((0)^6, (1)^3, 0)$, as:

$$\xi^{[3,2]} = 2(\Lambda^6 - \Lambda^7 - \Lambda^8) - \tilde{\xi}_\sigma^{[3,2]} = ((2)^6, 3, 3, 1, 2)\,. \tag{10.11}$$

[16]Note that both the signs of the orientifold plane charges and this angle χ would also be sensitive to the presence of a quantized Kalb-Ramond background flux $\int B_{89}\, dx'^8 dx^9$, but we neglect this possibility here, since it would be the sign of a tilted geometry in the $(x^8; x^9)$ plane on the type A side (non-trivial complex structure of the torus).

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Let us first blindly compute the ensuing mass formula, reduce it on x^{10} and T-dualize it on x^7:

$$M_p^{12} R_1 \cdots R_6 (R_7 R_8)^2 R_{10} \xrightarrow{M_p R_{10} \to 0} \frac{M_s^{11}}{g_A^3} R_1 \cdots R_6 (R_7 R_8)^2 \xrightarrow{\mathcal{T}_7} M_{\rm D9} = \frac{M_s^{10}}{g_B^3} R_1 \cdots R_7' R_8^2 \,. \tag{10.12}$$

On the type B side, R_8 and R_9 form the pair of orbifolded directions. Comparing with (10.10), we immediately see that we will have a D9-brane if: $R_9 \propto R_8/g_B^2$. As hinted above, we need to find an angle in the dual type 0A setup to identify the flux. In this perspective, we perform a further T-duality along x^8 that brings us to a $S^1/(-1)^{f_R}\Omega I_1$ orientifold of type 0A string theory in which the type 0B flux is mapped to an angle between the O8-plane, and the D8-brane obtained from (10.12) as

$$M_{\rm D9} = \frac{M_s^{10}}{g_B^3} R_1 \cdots R_7' R_8^2 \xrightarrow{\mathcal{T}_8} \frac{M_s^9}{g_A^3} R_1 \cdots R_6 R_7' R_8' \,. \tag{10.13}$$

This implies that there is a dual relation to $R_9 \propto R_8/g_B^2$ on the 0A side that has the same form:

$$R_9 \propto R_8/g_B^2 \xrightarrow{\mathcal{T}_8} R_9 \propto R_8'/g_A^2 \,.$$

Indeed, plugging back this dual relation in (10.13) clearly identifies the corresponding object with a D8-brane of type A string theory. Interestingly, (10.9) implies that there can be a non-right angle between the D8-brane and the O8-plane with $\cot(\chi) = \frac{c_1}{N}\frac{R_8}{R_9} \propto \frac{c_1}{N} g_A^2$. Unfortunately, our purely algebraic formalism does not allow us to see the individual values of c_1, N and the proportionality constant, but they must be physically chosen so that: $\frac{1}{2\pi}\int_{T^2/\mathbb{Z}_2} \mathrm{Tr}(F_{89}) dx^8 dx^9 = c_1 = 16$. This is similar to the case of [19], where the type of brane necessary for anomaly cancellation was obtained from the shift vector, but not their number.

Let us then study the case of a $T^2/\{(-1)^F S, \mathbb{Z}_2\}$ (where the shift operator S only acts on the M-theory direction x^{10}) orbifold of M-theory. We want to show that it gives an alternative M-theory lift of the same type 0' $T^2/(-1)^{f_R}\Omega I_2$ orientifold that we have just studied. Before we discuss the brane configuration, it is necessary to discuss the case of M-theory on $S^1/\{(-1)^F S, \mathbb{Z}_2\}$ to understand the effect of taking both orbifold projections on the same circle. We first remark that the orientifold group has four elements: $\{\mathbb{1}, (-1)^F S, I_1, (-1)^F I_1'\}$, where $I_1' = S I_1$. While I_1 is a reflexion of the coordinate x^{10} with respect to $x^{10} = 0$, I_1' is a reflection of x^{10} with respect to $x^{10} = \pi/2$. In particular, I_1 has two fixed points at $x^{10} = 0$ and π, while I_1' has two fixed points at $x^{10} = \pi/2$ and $3\pi/2$, and S has no fixed point. In particular, the fundamental domain is an interval $[0, \pi/2]$ and there are three types of twisted sectors, the usual bosonic closed string twisted sector of $(-1)^F S$ that leads to a type 0 spectrum and two open strings twisted sectors sitting at each pair of fixed points. What is not known, however, is the precise resulting gauge symmetry and twisted spectrum. There is a dual picture of the same model, where one first uses the S symmetry to reduce the circle by half, and then considers the projection by I_1 which replaces the circle by the interval. This second picture resembles the non-supersymmetric heterotic orbifold of M-theory discussed in [83], except that these authors did not include a closed string twisted sector, which hopefully helps stabilizing the non-supersymmetric theory. We now conjecture that M-theory on $T^2/\{(-1)^F S, \mathbb{Z}_2\}$ is the strong coupling limit of the $S^1/(-1)^{f_R}\Omega I_1$ orientifold of type 0A string theory and is thus T-dual to the $T^3/\mathbb{Z}_2 \times S^1/(-1)^F S$ orbifold of M-theory through a double T-duality, modulo the appropriate breaking of gauge groups by Wilson lines.

Let us be more concrete. We need to reduce to type 0A string theory on an orbifolded direction, then T-dualize to type 0B on a normal toroidal direction to reach a type 0' $T^2/(-1)^{f_R}\Omega I_2$ orientifold, as in the following mapping:

$$\begin{array}{lcl} \text{M-theory on } T^8 \times T^2/\{(-1)^F S, \mathbb{Z}_2\} & & \\ \quad\quad\quad\big\downarrow{\scriptstyle M_P R_{10}\to 0} & & \\ \text{type 0A on } T^8 \times S^1/(-1)^{f_R}\Omega I_1 & \xrightarrow{\mathcal{T}_8} & \text{type 0B on } T^7 \times T^2/(-1)^{f_R}\Omega I_2 \end{array} \tag{10.14}$$

First, we have to select one definite shift vector from all equivalent ones. We take the one that has the particularity to descend from the more general $T^2/\mathbb{Z}_n$ serie of shift vectors of the form $n\tilde{\delta}^{[2]} - \alpha_7$, namely:

$$\xi^{[2,2]} = 2(\Lambda^7 - \Lambda^8) + \Lambda^{\{2\}} = 2\tilde{\delta}^{[2]} - \tilde{\xi}^{[2,2]} = ((2)^8, 1, 1)\,, \tag{10.15}$$

which lead to the mass formulae:

$$M_p^{-9} V^{-1} e^{\langle \xi^{[2,2]}, H_R \rangle} = M_p^9 R_1 \cdots R_8 \xrightarrow{M_p R_{10}\to 0} \frac{M_s^9}{g_A^3} R_1 \cdots R_8 \xrightarrow{\mathcal{T}_8} \frac{M_s^{10}}{g_B^3} R_1 \cdots R_7 R_8'^2\,. \tag{10.16}$$

We immediately see that we end up with the same objects as in (10.12) and (10.13) and the analysis of fluxes and angles is completely parallel. In a sense, the presence of these two different M-theory lifts of the same string orientifold reflects the equivalence between T-dualizing $S^1/\Omega\mathbb{Z}_2$ in the transverse space and T-dualizing $T^3/\Omega\mathbb{Z}_2$ along an orbifold direction. We will use a similar property later to relate $\xi^{[6,2]}$ and $\xi^{[7,2]}$.

We can now turn to the $T^6/\{(-1)^F S, \mathbb{Z}_2\}$ orbifold of M-theory. We will study this case along the same line as $T^2/\mathbb{Z}_2$, first reducing on an orbifolded direction to a type 0A theory orientifold, then T-dualizing along a transverse direction to a type 0B orientifold:

$$\begin{array}{lcl} \text{M-theory on } T^4 \times T^6/\{(-1)^F S, \mathbb{Z}_2\} & & \\ \quad\quad\quad\big\downarrow{\scriptstyle M_p R_{10}\to 0} & & \\ \text{type 0A on } T^4 \times T^5/(-1)^{f_R}\Omega I_5 & \xrightarrow{\mathcal{T}_4} & \text{type 0B on } T^3 \times T^6/(-1)^{f_R}\Omega I_6 \end{array} \tag{10.17}$$

We will again have a system of N pairs of magnetized D9- and D9'-branes, now contributing to cancel the -1/4 units of negative D3-brane charge carried by each of the 64 O3-planes. This can be achieved by the Chern-Simons coupling:

$$\frac{M_s^4}{(2\pi)^3} \int_{\mathbb{R}\times T^3} C_4 \cdot \frac{1}{(2\pi)^3} \int_{T^6/\mathbb{Z}_2} \mathrm{Tr}(F_2 \wedge F_2 \wedge F_2)\,.$$

In the case of a factorizable metric, we can separate $T^6/\mathbb{Z}_2$ into 3 $T^2/\mathbb{Z}_2$ sub-orbifolds, and only F_{56}, F_{78} and F_{49} yield non-trivial fluxes. Instead of c_1 and N, we now introduce for each pair of coordinates $(x^i;x^j)$ of the T^2's pairs of quantized numbers denoted by (m^a_{ij},n^a_{ij}) [82]. The index a here numbers various stacks of N_a pairs of branes, with different fluxes. In the dual 0A picture, the m^a_{ij} and n^a_{ij}'s give wrapping numbers around the directions parallel, respectively perpendicular, to the O6-planes and a distinguishes between wrappings of branes around different homology cycles. With an appropriate normalization of cohomology bases on the homology cycles, one obtains:

$$\frac{1}{(2\pi)^3}\mathrm{Tr}\left(\int_{T^2/\mathbb{Z}_2} F_{56}\,dx^5dx^6\int_{T^2/\mathbb{Z}_2} F_{78}\,dx^7dx^8\int_{T^2/\mathbb{Z}_2} F_{49}\,dx'^4dx^9\right)=\sum_a N_a m^a_{56}m^a_{78}m^a_{49}=16$$

On the other hand, Chern-Simons couplings to higher forms such as C_5, C_7 and C_9 are determined by expressions which also include n^a_{ij} factors. For example, the D9-charge is related to $\sum_a N_a n^a_{56}n^a_{78}n^a_{49}$. The wrapping numbers should then be chosen in a way that all those other total charges cancel. There are in principle several ways to achieve this, but it is not our main focus, so we will not give a specific example here (see [84] for concrete realizations in the supersymmetric case). Rather, following the $T^6/\mathbb{Z}_2$ case above, one wishes to study the magnetized D9-brane action given by our algebraic method, deduce from it that certain pairs of radii are related and then perform a triple T-duality along $(x^4;x^6;x^7)$ to exchange the fluxes against tilting angles between O6-planes and pairs of D6-branes and their image D6'-branes.

Keeping this framework in mind, we first recall the choice of shift vector that comes from the general $T^6/\mathbb{Z}_n$ orbifold serie. It is given by:

$$\xi^{[6,2]}=2(\Lambda^7-\Lambda^8)+\tilde{\xi}^{[6,2]}=2\tilde{\delta}^{[6]}-\alpha_3+\alpha_5-\alpha_7=((2)^4,1,3,3,1,1,1)\,, \qquad (10.18)$$

where $\xi^{[6,2]}$ differs from its expression $\xi^{[6,n]}$ for $n=2$ given in Section 9.3 because of the charge Q_3 is now -1 instead of -2. Let us again follow the dualities (10.17) to see how the D9-brane is expressed in this formalism:

$$\frac{e^{\langle\xi^{[6,2]}|H_R\rangle}}{M_pV}=M_p^9R_1\cdot\cdot R_4(R_6R_7)^2\xrightarrow{M_pR_{10}\to 0}\frac{M_s^9}{g_A^3}R_1\cdot\cdot R_4(R_6R_7)^2\xrightarrow{\mathcal{T}_4}\frac{M_s^{10}}{g_B^3}R_1\cdot\cdot R_3(R_4'R_6R_7)^2\,.$$

This can match the action of a D9-brane if $R_5\propto R_6$, $R_7\propto R_8$ and $R_9\propto R_4'/g_B^2$. On the type A side, this again means that $R_9\propto R_4/g_A^2$, and one verifies easily that $\mathcal{T}_4$ indeed maps the D9-brane to a D8-brane extended along all directions except x^4. This D8-brane is tilted with respect to the O4-plane in the $(x^4;x^9)$-plane by an angle $\cot(\chi^a_{49})=\frac{m^a_{49}R_4}{n^a_{49}R_9}$ and still carries magnetic fluxes F_{56} and F_{78}. T-dualizing further along x^6 and x^7 leads to a D6-brane extended in the hypersurface along $(x^0;x^1;x^2;x^3;x^5;x^8;x^9)$ with mass:

$$\frac{M_s^7}{g_A}R_1\cdots R_4R_6R_7\sim\frac{M_s^7}{g_A^3}R_1\cdots R_3R_5R_8R_9\,.$$

Then, we can interpret this brane as one of the N_a D6-branes exhibiting two additional non-right angles with respect to the orientifold O6-plane, given by $\cot(\chi^a_{56})=\frac{m^a_{56}R_6}{n^a_{56}R_5}$ and $\cot(\chi^a_{78})=\frac{m^a_{78}R_7}{n^a_{78}R_8}$. It is of course understood in this discussion that the appropriate image Dp'-branes are always present.

Finally, we still wish to study $T^7/\{(-1)^F S, \mathbb{Z}_2\}$ orbifolds of M-theory. For this purpose, we use the permutation of $\tilde{\xi}^{[7,2]}$ describing an orbifold in $(x^3;\ldots;x^9)$ given by $\tilde{\xi}_\sigma^{[7,2]} = (0,0,(1)^7,2)$ in the following fashion:

$$\xi^{[7,2]} = 2(\Lambda^2 - \Lambda^3 + \Lambda^5 - 2\Lambda^8) + \tilde{\xi}_\sigma^{[7,2]} = (2,2,3,3,1,3,3,1,1,2).$$

This time, we follow the successive mappings

$$\begin{array}{c} \text{M-theory on } T^2 \times T^8/(-1)^F S \\ \Big\downarrow {\scriptstyle M_p R_{10} \to 0} \\ \text{type 0A on } T^2 \times T^7/(-1)^{f_L}\Omega I_7 \xrightarrow{\mathcal{T}_3} \text{type 0B on } T^3 \times T^6/(-1)^{f_R}\Omega I_6 \end{array}$$

leading to the mass formulae:

$$M_p^{12} R_1 R_2 (R_3 R_4 R_6 R_7)^2 R_{10} \xrightarrow{M_p R_{10}} \frac{M_s^{11}}{g_A^3} R_1 R_2 (R_3 R_4 R_6 R_7)^2 \xrightarrow{\mathcal{T}_3} \frac{M_s^{10}}{g_B^3} R_1 \cdots R_3' (R_4 R_6 R_7)^2 \,.$$

and we obtain again the same type 0B $T^6/(-1)^{f_R}\Omega I_6$ orientifold as above, while tilting angles in the dual type IIA picture can again be obtained by $\mathcal{T}_{467}$.

Overall, we have a fairly homogeneous approach to these four different orbifolds of M-theory and it should not be too surprising that their untwisted sectors build the same algebra. We finally summarize the shift vectors we used for physical interpretation in Table 20. It is remarkable that these roots are found at level 6 and 7 in α_8, showing again that a knowledge of the $\mathfrak{e}_{10}$ root space at high levels is essential for the algebraic study of M-theory orbifolds.

Another fact worth mentioning is that our $\mathbb{Z}_2$ shift vectors either have norm 2 or -2, in contrast to the null shift vectors of Section 10.1. This lightlike characteristic has been proposed in [21, 19] to be a general algebraic property characterizing Minkowskian branes in M-theory. Similarly, these authors associated instantons with real roots of $\mathfrak{e}_{10}$, viewed as extensions of roots of $\mathfrak{e}_8$, that all have norm 2. However, we have just shown that Minkowskian objects can just as well have norm 2, or -2, and perhaps almost any. We suggest that the deciding factor is the threshold rather than the norm (at least for objects coupling to forms, forgetting for a while the exceptional case of Kaluza-Klein particles that have negative threshold, when they are instantonic and null threshold, when they are Minkowskian). Indeed, instantonic objects have threshold 0, while Minkowskian ones have threshold 1. This approach is compatible with the point of view of [18], as explained in Section 3.5.2, as well as with the results of this subsection. Some higher threshold roots also appear in Table 14 and 15, however, but we leave their interpretation for further investigation.

11. Shift vectors for $\mathbb{Z}_n$ orbifolds: an interpretative prospect

Now that we have an apparently coherent framework to treat $\mathbb{Z}_2$ M-theory orbifolds, it is tempting to try to generalize it to all $\mathbb{Z}_n$ orbifolds. To understand how this could be done, it is instructive to look at Tables 14 and 15. As mentioned at the end of Section 9.3, one notices that shift vectors for $T^q/\mathbb{Z}_n$ orbifolds can typically be grouped in series, for successive values of q and n. As an illustration, we give one such serie (i.e. relating orbifolds with all charges ± 1) in the following table:

$n \backslash q$	2	4	6	8	10
2	$((2)^8, 1, 1)$	$((2)^6, 3, 1, 1, 1)$	$((2)^4, 1, 3, 3, 1, 1, 1)$	$(2, 2, 3, 1, 1, 3, 3, 1, 1, 1)$	/
3	$((3)^8, 2, 1)$	$((3)^6, 4, 2, 2, 1)$	$((3)^4, 2, 4, 4, 2, 2, 1)$	$(3, 3, 4, 2, 2, 4, 4, 2, 2, 1)$	$(2, 4, 4, 2, 2, 4, 4, 2, 2, 1)$
4	$((4)^8, 3, 1)$	$((4)^6, 5, 3, 3, 1)$	$((4)^4, 3, 5, 5, 3, 3, 1)$	$(4, 4, 5, 3, 3, 5, 5, 3, 3, 1)$	$(3, 5, 5, 3, 3, 5, 5, 3, 3, 1)$
5	$((5)^8, 4, 1)$	$((5)^6, 6, 4, 4, 1)$	$((5)^4, 4, 6, 6, 4, 4, 1)$	$(5, 5, 6, 4, 4, 6, 6, 4, 4, 1)$	$(4, 6, 6, 4, 4, 6, 6, 4, 4, 1)$
6	$((6)^8, 5, 1)$	$((6)^6, 7, 5, 5, 1)$	$((6)^4, 5, 7, 7, 5, 5, 1)$	$(6, 6, 7, 5, 5, 7, 7, 5, 5, 1)$	$(5, 7, 7, 5, 5, 7, 7, 5, 5, 1)$

From this table, it should be immediately apparent that typical shift vectors for $T^q/\mathbb{Z}_n$ orbifolds, with $q \in 2\mathbb{N}$ are given by (some permutation of):

$$\begin{aligned} \xi &= n\tilde{\delta} + \sum_{i=1}^{q/2} (-1)^{q/2-i}\, p_i\, \alpha_{7-q+2i} \\ &= ((n)^{10-q}, n + p_1, n - p_1, n - p_2, n + p_2, \dots, n - p_{q/2}, p_{q/2}) \end{aligned}$$

and have a threshold bigger or equal to 1 since $1 \le q_i \le n-1$, $\forall i = 1, \dots, q/2$. In analogy with the $\mathbb{Z}_2$ orientifold cases, it is tempting to think of the "average" value $((n)^9, 0)$ as spacetime-filling branes, and of the deviations $q_i \alpha_{7-q+2i}$ as fluxes in successive pairs of (orbifolded) dimensions. Of course, the fluxes are only directly interpretable as such after the reduction to string theory. In the $\mathbb{Z}_2$ examples, they appeared because an M-theory orbifold turns into a string theory orientifold with open strings twisted sectors exhibiting non-abelian Chan-Paton factors. This allowed us to invoke Chern-Simons couplings of the form:

$$\int C_{10-q} \cdot \int \mathrm{Tr}(F^{q/2}) \tag{11.1}$$

on the world-volume of the space-filling branes that participate to tadpoles cancellation at the orbifold fixed points. Geometrically speaking, the more orbifolded directions, the more non-trivial fluxes can be switched on, producing higher non-zero Chern numbers that reflect the increasingly complex topology in the presence of several conifold singularities at each fixed point. A further research direction is to determine which kind of flux could appear in which $\mathbb{Z}_n$ orbifolds.

In any case, one should not forget that the orbifolded directions in the string theory limit are not exactly the same as in the original M-theory orbifold, so that a bit of caution is required when trying to interpret the shift vector directly, without going through a chain of dualities leading to a better-known string theory soliton.

Our proposal is to regard the mass formulae associated to these shift vectors as M-theory lifts of the resulting string theory brane configurations, that are somehow necessary for the M-theory orbifolds to be well-defined, in a sense which remains to be understood.

It also remains unclear how the change of average value of the components of the shift vector from 2 to n determines the fact that we have a higher order orbifold. Intuitively, it should reflect the presence of more twisted sectors, but is a priori not related to the different number of fixed points.

All these questions are of course of primary interest to obtain non-trivial physical information from our algebraic toolkit and we will pursue them in forthcoming research projects. They will be addressed in future publications.

12. Conclusion

In this paper, we have aimed at developing a rigorous and general algebraic procedure to study orbifolds of supergravity theories using their U-duality symmetry. We were particularly interested in the $\mathfrak{e}_{11-D|11-D}$ serie of real split U-duality algebras for $D = 1, \ldots, 8$. Essentially, the procedure can be decomposed in the following successive steps. First, one constructs a finite order non-Cartan preserving inner automorphism describing the orbifold action in the complexified algebra $\mathfrak{e}_{11-D}$. This nth-order rotation automorphism reproduces the correct $\mathbb{Z}_n$-charges of the physical states of the theory, when using the "duality" mapping relating supergravity fields and directions in the coset $\mathfrak{e}_{11-D|11-D}/\mathfrak{k}(\mathfrak{e}_{11-D|11-D})$ (in the symmetric gauge). Next, one derives the complexified invariant subalgebra satisfied by the null charge sector and fixes its real properties by taking its fixed point subalgebra under the restricted conjugation. One then moves to an eigenbasis, on which the orbifold action takes the form of a Cartan-preserving (or chief) inner automorphism, and computes, in terms of weights, the classes of shift vectors reproducing the expected orbifold charges for all root spaces of $\mathfrak{e}_{11-D}$. In $D = 1$, one uses the invariance modulo n to show that every such class contains a root of $\mathfrak{e}_{10}$, which can be used as the class representative. In a number of cases, these roots can be identified with Minkowskian objects of M-theory or of the lower-dimensional string theories, and interpreted as brane configurations necessary for anomaly cancellation in the corresponding orbifold/orientifold setups.

In fact, for a given $T^q/\mathbb{Z}_n$ orbifold, the first two steps only have to be carried out explicitly once in $\mathfrak{e}_{q+1}$ for the compactification space $S^1 \times T^q/\mathbb{Z}_n$, and need not be repeated for all $T^p \times T^q/\mathbb{Z}_n$. Rather, one can deduce in which way the Dynkin diagram of the invariant subalgebra will get extended upon further compactifications. This is relatively straightforward until $D = 3$, but requires some more care in $D = 2, 1$, when the U-duality algebra becomes infinite-dimensional. In $\mathfrak{e}_{10}$, in particular, a complete determination of the root system of the invariant subalgebra requires in principle to look for all invariant generators. This could in theory be done, provided we know the full decomposition of $\mathfrak{e}_{10}$ in representations of $\mathfrak{sl}(10, \mathbb{R})$. However, one of the conclusions of our analysis is that once we understand the structure of $\mathfrak{g}_{\text{inv}}$ at low-level, its complete root system can be inferred from the general structure of Borcherds algebras.

By doing so, however, one realizes that there are three qualitatively different possible situations from which all cases can be inferred. The determining factor is the invariant subalgebra in $D = 3$. If this subalgebra of $\mathfrak{e}_{8|8}$ is simple, its extension in $\mathfrak{e}_{10|10}$ is hyperbolic and non-degenerate. This happens for $T^q/\mathbb{Z}_2$ for $q = 1, 4, 5, 8, 9$, as already shown in [19] by alternative methods. If it is on the other hand semi-simple, we obtain, in $D = 2$, what we called an affine central product. It denotes a product of the affinization of all simple factors present in $D = 3$, in which the respective centres and derivations of all factors are identified. Descending to $D = 1$, all affine factors reconnect through α_{-1} in a simple Dynkin diagram, leading to a degenerate hyperbolic Kac-Moody algebra, but without its natural centre(s) and derivation(s). This is the case for all remaining $\mathbb{Z}_2$ orbifolds, as well as for $T^6/\mathbb{Z}_n$ orbifolds with $n = 3, 4$. Finally, if an abelian factor is present in $\mathfrak{e}_8$, its affinization in $\mathfrak{e}_9$, $\hat{\mathfrak{u}}(1)$, turns into all multiples of an imaginary root in $\mathfrak{e}_{10}$, which also connects through α_{-1} to the main diagram, thus leading to a Borcherds algebra with one isotropic simple root. Although it was conceptually clear to mathematicians that Borcherds algebras can emerge as fixed-point subalgebras of Kac-Moody algebras under automorphisms, we found here several explicit constructions, demonstrating how this comes about in examples of a kind that does not seem to appear in the mathematical literature.

In the first case, the multiplicity of invariant roots is inherited from $\mathfrak{e}_{10}$, in the other two cases, however, great care should be taken in understanding how the original multiplicities split between different root spaces. In fact, the Borcherds/indefinite KM algebras appearing in these cases provide first examples of a splitting of multiplicities of the original KMA into multiplicities of several roots of its fixed point subalgebra. This is strictly speaking the case only for the algebras as specified by their Dynkin diagram, but one should keep in mind that the quotient by its possible derivations suppresses the operators that could differentiate between these roots, and recombines them into root spaces of the original dimension, albeit with a certain redistribution of the generators. In fact, it is likely that a computation of the root multiplicities by an appropriate Kac-Weyl formula for GKMA based on the root system of the Dynkin diagram would predict slightly smaller root spaces than those of the fixed-point subalgebra that are obtained from our method. However, it is not absolutely clear what is the right procedure to compute root multiplicities in GKMA. This is a still largely open question in pure mathematics, on which our method will hopefully shed some light.

Along the way, we also explicitly showed, in the $T^4/\mathbb{Z}_n$ case, how to go from our completely real basis for $\mathfrak{g}_{\text{inv}}$, described by a fixed-point subalgebra under the restricted conjugation, to the standard basis of its real form, obtained from the Cartan decomposition. This is especially interesting in the affine case, where we obtained the relation between the two affine parameters and their associate derivations.

Even though the present paper was focused on the breakings of U-duality symmetries, it is clear that, in another perspective, the same method can in principle be applied to obtain the known classification of (symmetric) breaking patterns of the $E_8 \times E_8$ gauge symmetry of heterotic string theory (or any other gauge symmetry) by orbifold projections. Indeed, our result in $D = 3$ for breakings of $\mathfrak{e}_8$ can be found in the tables of [27, 28], where they are derived from the Kac-Peterson method using chief inner automorphisms. Reciprocally, one might wonder why we did not use the Kac-Peterson method to study U-duality symmetry, too. It is certainly a beautiful and simple technique, very well suited to classify all possible non-isomorphic symmetry breakings of one group by various orbifold actions. However, calculating with $\mathbb{Z}_n$-rotation automorphisms instead of Cartan-preserving ones has a number of advantages when dealing with U-duality symmetries. In the Kac-Peterson method, one first fixes n, then lists all shift vectors satisfying the condition $(\Lambda, \theta_G) \leq n$ of

Section 9, which allows to obtain all non-isomorphic breakings. In the end, however, one has sometimes to resort to different techniques to associate these breakings with a certain orbifold with determined dimension and charges.

Here, we adopt a quite opposite philosophy, by resorting to non-Cartan preserving inner automorphisms with a clear geometrical interpretation. In this perspective, one starts by fixing the dimension and charges of the orbifold and then computes the corresponding symmetry breaking, which allows to discriminate easily between a degenerate finite order rotation and an effective one. Only then do we reexpress this automorphism in an eigenbasis of the orbifold action, in which it takes the form of a chief inner automorphism, and compute the class of associated shift vectors. Doing so, we can unambiguously assign a particular class of shift vectors to a definite orbifold projection in space-time. Note that such shift vectors will typically not satisfy $(\Lambda, \theta_G) \leq n$, so that a further change of basis is required to relate them to their conjugate shift vector in the Kac-Peterson formalism (we have shown in Section 9 how to perform this change of basis explicitly). However, this process may obscure the number of orbifolded dimensions and the charge assignment on the Kac-Peterson side.

Furthermore, another reason for not resorting to the Kac-Peterson method is that we are not so much interested in all possible breakings of one particular group, say E_8, as in determining the fixed-point subalgebras for the whole E_r serie. Consequently, we can concentrate on the $T^q/\mathbb{Z}_n$ orbifold action in E_{q+1} and then extend the result to the whole serie without too many additional computations, since the orbifold rotation acts trivially on the additional compactified dimension and the natural geometrical interpretation of the $SL(r,\mathbb{R}) \in E_r$ generators has been preserved. On the other hand, the change of basis necessary to obtain a shift vector satisfying the Kac-Peterson condition can be completely different in E_r compared to E_{r-1}. Accordingly, starting from such a shift vector for E_{r-1}, there is no obvious way to obtain its extension describing the same orbifold in E_r. Finally, and much more important to us is the fact that there is no known way to extend the Kac-Peterson method to the infinite-dimensional case.

The above discussion has concentrated on the part of this work where the invariant subalgebras of $\mathfrak{e}_{11-D|11-D}$ under an nth-order inner automorphism were derived. In $D \geq 3$, these describe the residual U-duality symmetry and bosonic spectrum of supergravity theories compactified on orbifolds and map to the massless bosonic spectrum of the untwisted sector of orbifolded string theories. In such cases, these results have been known for a long time. They are however new in $D = 2, 1$, which was the main focus of this research project. In particular, the $D = 1$ case is very interesting, since the hyperbolic U-duality symmetry encountered there is expected to contain non-perturbative information, as well. Indeed, the specific class representatives of shift vectors we find correspond to higher level roots of $\mathfrak{e}_{10}$ which have no direct interpretation as supergravity fields. It is thus tempting to try to relate them to non-classical effects in M-theory which might give us information on the twisted sector of orbifolds/orientifolds of the descendant string theories.

Let us now discuss this more physical interpretative aspect inspired from the work of [19], where the shift vectors for a restricted class of $\mathbb{Z}_2$ orbifolds of M-theory were shown to reproduce the mass formulae of Minkowskian branes, which turned out to be the correct objects to be placed at each orbifold fixed point to ensure anomaly/tadpole

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cancellation. We have extended this analysis to incorporate other $\mathbb{Z}_2$ orbifolds of M-theory, which are non-supersymmetric and should be considered in bosonic M-theory. They have the particularity to break the infinite U-duality algebra to indefinite KMAs. These orbifolds reduce to $T^2/(-1)^{f_R}\Omega\mathbb{Z}_2$ and $T^6/(-1)^{f_R}\Omega\mathbb{Z}_2$ orientifolds of the type 0B string theory in which pairs of magnetized D9- and D9'-branes are used to cancel the O7- (resp. O3-) plane charges. They are part of a chain of dual orientifolds starting from type 0B string theory on $(-1)^{f_L}\Omega$, a tachyon-free theory believed to be well-defined, usually referred to as type 0' string theory. We have then shown that the $\mathfrak{e}_{10}$ roots playing the rôle of class representatives of shift vectors in these cases can be interpreted as such space-time filling D9-branes carrying the appropriate configuration of magnetic fluxes. This identification could in turn serve as a proposal for M-theory lifts of such type 0B orientifolds, as generated by certain exotic objects corresponding to $\mathfrak{e}_{10}$ roots that are not in $\mathfrak{e}_9$. Finally, these type IIB setups have an alternative reading in the T-dual type IIA pictures where the magnetic fluxes appear as tilting angles between O8- (resp. O6-)planes and D8- (resp. D6-)branes and their image branes, our analysis providing an algebraic characterization of this tilting angle.

As for $\mathbb{Z}_{n\geq 3}$ case, even though we have treated only a few examples explicitly, we have noticed that their associated shift vectors fall into series of roots of $\mathfrak{e}_{10}$, for successive values of q and n, with remarkable regularity. This has provided us with a facilitated procedure for constructing shift vectors for any $T^q/\mathbb{Z}_n$ orbifold which acts separately on each of the $(q/2)$ T^2 subtori. These roots of level $3n$ are classified in Tables 14 and 15. Despite the remarkably regular structure of such roots, it is not completely clear how to extract information on the correct anomaly/tadpole-cancelling brane configurations of the corresponding orbifolds. In particular, the components of the shift vectors transverse to the orbifold increase monotonously with n, so that their interpretation requires novel ideas. However, it is clearly of interest to generalize the identification of such brane constructions for $\mathbb{Z}_n$-shift vectors with $n > 2$, and to understand their possible relation to twisted sectors and/or fluxes present in the related string orbifolds. Hopefully, this can be done in a systematic manner, reproducing what is known about string theory orbifolds/orientifolds and leading to predictive results about less-known types of M-theory constructions.

Another future direction of research would consist in investigating more complicated orbifold setups in our algebraic framework, in which, for instance, several projections of various orders are acting on the same directions. This could possibly lead to new interesting classes of GKMAs. In general, however, not only one, but two or more shift vectors will be necessary to generate such orbifolds and should from a physical perspective be interpreted separately. This will hopefully open the door to working out the physical identification of yet a larger part of the $\mathfrak{e}_{10}$ root system, and constitute another step in the understanding of the precise relation between M-theory and $\mathfrak{e}_{10}$.

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A. Highest roots, weights and the Matrix R

i) The matrix R: herebelow, we give the expression of the matrix R used in Section 3.1 to define the root lattice metric g_ε (3.6) in the physical basis:

$$R = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/3 & -2/3 & 1/3 \end{pmatrix} .$$

ii) Highest roots of the exceptional E_r chain: We list the highest roots of the finite Lie algebras of the chain $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \ldots \subset \mathfrak{a}_4 \subset \mathfrak{d}_5 \subset \mathfrak{e}_6 \subset \mathfrak{e}_7 \subset \mathfrak{e}_8$, appearing throughout this article:

$$\begin{aligned} \theta_{A_1} &= \alpha_8 \,, \\ \theta_{A_i} &= \alpha_{8-i} + .. + \alpha_7 \,, \quad i = 2, 3 \\ \theta_{A_4} &= \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 \,, \\ \theta_{D_5} &= \alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8 \,, \\ \theta_{E_6} &= \alpha_3 + 2\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + 2\alpha_8 \,, \\ \theta_{E_7} &= \alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + 2\alpha_8 \,, \\ \theta_{E_8} &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8 \,. \end{aligned}$$

ii) Fundamental weights of $\mathfrak{e}_{10}$: The expression, on the set of simple roots, of the fundamental weights of $\mathfrak{e}_{10}$ defined by $(\Lambda^i|\alpha_j) = \delta^i_j$ for $i, j = -1, 0, 1, .., 8$ is obtained by inverting $\Lambda^i = (A(\mathfrak{e}_{10})^{-1})^{ij}\alpha_j$. In the physical basis, these weights have the particularly simple expression:

	$\lvert\Lambda\rvert^2$		$\lvert\Lambda\rvert^2$
$-\Lambda^{-1} = (0,1,1,1,1,1,1,1,1,1)$	0	$-\Lambda^4 = (5,5,5,5,5,5,6,6,6,6)$	-30
$-\Lambda^0 = (1,1,2,2,2,2,2,2,2,2)$	-2	$-\Lambda^5 = (6,6,6,6,6,6,6,7,7,7)$	-42
$-\Lambda^1 = (2,2,2,3,3,3,3,3,3,3)$	-6	$-\Lambda^6 = (4,4,4,4,4,4,4,4,5,5)$	-18
$-\Lambda^2 = (3,3,3,3,4,4,4,4,4,4)$	-12	$-\Lambda^7 = (2,2,2,2,2,2,2,2,2,3)$	-4
$-\Lambda^3 = (4,4,4,4,4,5,5,5,5,5)$	-20	$-\Lambda^8 = (3,3,3,3,3,3,3,3,3,3)$	-10

For their expression in the root basis, see, for instance, [85]. It can be recast in the following recursion relations:

$$\begin{aligned}
\Lambda^{-1} &= -\delta\,, & \Lambda^4 &= 2\Lambda^3 - \Lambda^2 - \alpha_3\,,\\
\Lambda^0 &= -(\alpha_{-1} + 2\delta)\,, & \Lambda^5 &= 2\Lambda^4 - \Lambda^3 - \alpha_4\,,\\
\Lambda^1 &= 2\Lambda^0 + \theta_{E_8}\,, & \Lambda^6 &= \Lambda^3 + \theta_{D_5}\,,\\
\Lambda^2 &= 2\Lambda^1 - \Lambda^0 - \alpha_1\,, & \Lambda^7 &= 2\Lambda^6 - \Lambda^5 - \alpha_6\,,\\
\Lambda^3 &= 2\Lambda^2 - \Lambda^1 - \alpha_2\,, & \Lambda^8 &= \Lambda^2 + \theta_{E_6}\,.
\end{aligned}$$

B. The U-duality group for 11D supergravity

This appendix is meant as a complement to Section 3.4, and reviews the U-duality transformations for finite U-duality groups as presented in [23], which trivially extends to the hyperbolic case in $D = 1$.

It has been shown in [86, 11, 22], that the $E_{11-D|11-D}(\mathbb{Z})$ chain of U-duality group relevant for M-theory incorporates a generalized T-duality symmetry, which exchanges not only the radii of 10-dimensional IIA theory among themselves, but also the 11-dimensional radius R_{10} with any of them, leading to the transformation:

$$\mathcal{T}_{ijk} : R_i \to \frac{1}{M_P^3 R_j R_k}\,, \quad R_j \to \frac{1}{M_P^3 R_i R_k}\,, \quad R_k \to \frac{1}{M_P^3 R_i R_j}\,, \quad M_P^3 \to M_P^6 R_i R_j R_k\,. \tag{B.1}$$

for $i, j, k \in \{D, .., 10\}$. To get the whole Weyl group of E_{11-D}, one must supplement the transformation (B.1) with the permutation of all radii (belonging to the $SL(11-D,\mathbb{Z})$ modular group of the torus)

$$\mathcal{S}_{ij} : R_i \leftrightarrow R_j\,,$$

which is part of the permutation group $\mathcal{S}_{11-D}$ generated by the neighbour to neighbour permutations $\{S_{i,i+1}\}_{i=D,..,9}$. Then, taking the closure of the latter with the generator $\mathcal{T}_{89\,10}$ leads to the Weyl group:

$$W(E_{11-D}) = \mathbb{Z}_2 \overline{\ltimes} \mathcal{S}_{11-D} \tag{B.2}$$

with $\mathbb{Z}_2 = \{\mathbb{1}, \mathcal{T}_{8\,9\,10}\}$. This gives the whole set of Weyl generators in terms of their action on the M-theory radii.

If we compactify to IIA string theory by setting $M_P R_{10} \to 0$, then the generators

$$\mathcal{T}_{\hat{\imath}\hat{\jmath}10} : R_{\hat{\imath}} \to \frac{1}{M_s^2 R_{\hat{\jmath}}}\,, \quad R_{\hat{\jmath}} \to \frac{1}{M_s^2 R_{\hat{\imath}}}\,, \quad g_A \to \frac{g_A}{M_s^2 R_{\hat{\imath}} R_{\hat{\jmath}}}$$

for $\hat{\imath}, \hat{\jmath} \in \{D, .., 9\}$, represent a double T-duality symmetry mapping IIA string theory to itself. Likewise, the group of permutations is reduced to $\mathcal{S}_{10-D}$, generated by $\{S_{\hat{\imath},\hat{\imath}+1}\}_{\hat{\imath}=D,..,8}$, which belong to the $SL(10-D,\mathbb{Z})$ modular group of the IIA torus.

In $D = 1$, this setup naturally extends to the dilaton vector $H_R \in \mathfrak{h}(E_{10})$. The permutation group $\mathcal{S}_{10}$ acts as $H_R^i \to H_R^j$, for $i, j = 1, .., 10$, which corresponds to the dual Weyl transformation: $r_\alpha^\vee(H_R) = H_R - \langle H_R, \alpha\rangle \alpha^\vee$ for $\alpha = \alpha_{i-2} + \ldots + \alpha_{j-3} \in \Pi(A_9)$.

The $\mathbb{Z}_2$ factor in expression (B.2) on the other hand, corresponds to a Weyl reflection with respect to the electric coroot:

$$\begin{aligned} r_8^\vee(H_R) &= H_R - \langle H_R, \alpha_8\rangle \alpha_8^\vee \\ &= \left(H_R^1 + \frac{1}{3}\Delta H, H_R^2 + \frac{1}{3}\Delta H, .., H_R^7 + \frac{1}{3}\Delta H, H_R^8 - \frac{2}{3}\Delta H, H_R^9 - \frac{2}{3}\Delta H, H_R^{10} - \frac{2}{3}\Delta H\right)\,, \end{aligned}$$

with $\Delta H = H_R^8 + H_R^9 + H_R^{10}$.

On the generators of $\mathfrak{e}_{10}$, the Weyl group will act as $\sigma_\alpha = \exp\left[\frac{\pi i}{2}(E_\alpha + F_\alpha)\right]$ or alternatively as $\tilde{\sigma}_\alpha = \exp\left[\frac{\pi}{2}(E_\alpha - F_\alpha)\right]$, $\forall \alpha \in \Delta_+(E_{10})$, depending on the choice of real basis. In particular, a $\mathbb{Z}_4$ orbifold of M-theory can be represented in our language by a Weyl reflection, and is thus naturally incorporated in the U-duality group.

As mentioned in Section 3.4, from the point of view of its moduli space, the effect of acting with the subgroup $W(E_{11-D})$ of the U-duality group on the objects of M-theory on T^{10} will typically be to exchange instantons which shift fluxes, with instantons that induce topological changes. On the cosmological billiard, a Weyl transformation will then exchange the corresponding walls among themselves.

The rest of the U-duality group is given by the Borel generators. These act on the expectation values $\mathcal{C}_\alpha$, $\alpha \in \Delta_+(E_{11-D})$, appearing, in particular, as fluxes in expression (3.51). Picking, in a given basis, a root $\beta \in \Delta_+(E_{11-D})$, its corresponding Borel generator B_β will act on the (infinite) set $\{\mathcal{C}_\alpha\}_{\alpha \in \Delta_+(E_{11-D})}$ typically as [23, 63]:

$$B_\beta : \quad \mathcal{C}_\beta \to \mathcal{C}_\beta + 1 \qquad \mathcal{C}_\gamma \to \mathcal{C}_\gamma + \mathcal{C}_{\gamma-\beta}\,, \text{ if } \gamma - \beta \in \Delta_+(E_{11-D})\,. \tag{B.3}$$

If $\gamma - \beta \notin \Delta_+(E_{11-D})$, then $B_\beta : \mathcal{C}_\gamma \to \mathcal{C}_\gamma$. The first transformation in eqn.(B.3) is the M-theory spectral flow [23], generated by part of the Borel subalgebra of the arithmetic group $E_{11-D}(\mathbb{Z})$. Invariance under such a unity shift reflects the periodicity of the expectation values of the fields $\mathcal{A}^i{}_1$, C_3, $\widetilde{C}_6$ and $\widetilde{\mathcal{A}}^i{}_7$.

C. Conventions and involutive automorphisms for the real form $\widehat{\mathfrak{so}}(8,6)$

i) *Conventions* for $\mathfrak{d}_7$: we recall the conventions used in Section 6.3 to label the basis of simple roots of the finite $\mathfrak{d}_7 \subset \hat{\mathfrak{d}}_7 \subset \mathfrak{g}_{\rm inv}$ Lie algebra for the $T^5 \times T^4/\mathbb{Z}_{n>2}$ orbifold of

M-theory:

$$\beta_1 \equiv \alpha_- \,, \quad \beta_2 \equiv \tilde{\alpha} \,, \quad \beta_3 \equiv \alpha_+ \,, \quad \beta_4 \equiv \alpha_3 \,, \quad \beta_5 \equiv \alpha_2 \,, \quad \beta_6 \equiv \alpha_1 \,, \quad \beta_7 \equiv \gamma \,. \tag{C.1}$$

The affine $\widehat{\mathfrak{d}}_r$ will be described by the following Dynkin diagram: The lexicographic order

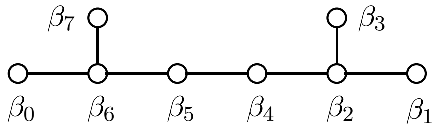

Figure 4: Dynkin diagram of $\hat{\mathfrak{d}}_7$ in the β-basis

used in convention (C.1) is meant to naturally extend the $\mathfrak{a}_3 \subset \mathfrak{g}_{\text{inv}}$ subalgebra appearing for the $T^4/\mathbb{Z}_{n>2}$ orbifold in $D=5$. In particular, we define $E_{\underline{i\cdots 4123}} \doteq [E_{\underline{i}}, \ldots [E_{\underline{4}}, E_{\alpha_- + \tilde{\alpha} + \alpha_+}]] \ldots]$ for $i \geqslant 4$.

For non-simple roots of level 2 in β_2, the corresponding ladder operator is defined by commuting two successive layers of simple root ladder operators, as, for instance, in:

$$E_{\underline{765^2 4^2 12^2 3}} \doteq [E_{\underline{5}}, [E_{\underline{4}}, [E_{\underline{2}}, E_{\underline{7654123}}]]] \,.$$

This implies in particular the useful relation

$$\mathcal{N}_{\beta_i, \beta_{i-1\cdots j^2 (j-1)^2 \cdots 4^2 12^2 3}} = \mathcal{N}_{\beta_j, \beta_{i\cdots j (j-1)^2 \cdots 4^2 12^2 3}} \,,$$

which, combined with $\mathcal{N}_{-\alpha,-\beta} = -\mathcal{N}_{\alpha,\beta}$ and $\mathcal{N}_{\alpha,\gamma-\alpha} = -\mathcal{N}_{\alpha,-\gamma}$, induces

$$\begin{aligned} \Big[F_{\underline{i}}, E_{\underline{i\cdots(j+1)j^2\cdots 4^2 12^2 3}}\Big] &= E_{\underline{i-1\cdots(j+1)j^2\cdots 4^2 12^2 3}} \,, \\ \Big[F_{\underline{j}}, E_{\underline{i\cdots(j+1),j^2\cdots 4^2 12^2 3}}\Big] &= E_{\underline{i\cdots j+1, j(j-1)^2\cdots 4^2 12^2 3}} \,. \end{aligned}$$

ii) <u>*The representation* Γ</u>: the inner involutive automorphism written in the form (6.21) acts on elements of the algebra $\mathfrak{d}_7$ in the representation $\Gamma\{1,0,\ldots,0\}$ (see [65]) defined as follows. For general r, let the basis of simple roots $\mathfrak{d}_r$ characterized by the Dynkin diagram of Figure 4 be recast in terms of the orthogonal basis ε_i, $i = 1,..,r$

$$\begin{gathered} \beta_1 = \varepsilon_{r-1} - \varepsilon_r \,, \qquad \beta_2 = \varepsilon_{r-2} - \varepsilon_{r-1} \,, \qquad \beta_3 = \varepsilon_{r-1} + \varepsilon_r \,, \\ \beta_i = \varepsilon_{r+1-i} - \varepsilon_{r+2-i} \,, \quad \forall i = 4, \ldots, r \,. \end{gathered} \tag{C.2}$$

The remaining non-simple roots can be reexpressed as follows: for $1 \leqslant i < j \leqslant r-3$, we have

$$\begin{gathered} \beta_{r+1-i} + \ldots + \beta_{r+1-j} = \varepsilon_i - \varepsilon_{j+1} \,, \\ \beta_{r+1-i} + \ldots + \beta_4 + \beta_2 = \varepsilon_i - \varepsilon_{r-1} \,, \\ \begin{aligned} \beta_{r+1-i} + \ldots + \beta_4 + \beta_2 + \beta_1 &= \varepsilon_i - \varepsilon_r \,, & \beta_2 + \beta_1 &= \varepsilon_{r-2} - \varepsilon_r \,, \\ \beta_{r+1-i} + \ldots + \beta_4 + \beta_3 + \beta_2 &= \varepsilon_i + \varepsilon_r \,, & \beta_3 + \beta_2 &= \varepsilon_{r-2} + \varepsilon_r \,, \\ \beta_{r+1-i} + \ldots + \beta_4 + \beta_3 + \beta_2 + \beta_1 &= \varepsilon_i + \varepsilon_{r-1} \,, & \beta_3 + \beta_2 + \beta_2 &= \varepsilon_{r-2} + \varepsilon_{r-1} \,, \end{aligned} \end{gathered} \tag{C.3}$$

while roots of level 2 in β_2 decompose as

$$\begin{aligned}\beta_{r+1-i}+\ldots+\beta_{r+1-j}+2(\beta_{r-j}+\ldots+\beta_4+\beta_2)+\beta_3+\beta_1 &= \varepsilon_i+\varepsilon_{j+1}\,,\; 1\leqslant i<j\leqslant r-4\,,\\ \beta_{r+1-i}+\ldots+\beta_4+2\beta_2+\beta_3+\beta_1 &= \varepsilon_i+\varepsilon_{r-2}\,,\; 1\leqslant i\leqslant r-4\,.\end{aligned}$$

Introducing the elementary matrices $\mathcal{E}_{i,j}$, with components $(\mathcal{E}_{i,j})_{kl}=\delta_{ik}\delta_{jl}$, the Cartan subalgebra of $\mathfrak{d}_r$ may be cast in the form

$$\begin{aligned}\Gamma(H_{\underline{1}}) &= \frac{1}{\sqrt{r(r-1)}}\left(\mathcal{E}_{r-1,r-1}-\mathcal{E}_{r,r}+\mathcal{E}_{r+1,r+1}-\mathcal{E}_{r+2,r+2}\right)\,,\\ \Gamma(H_{\underline{2}}) &= \frac{1}{\sqrt{r(r-1)}}\left(\mathcal{E}_{r-2,r-2}-\mathcal{E}_{r-1,r-1}+\mathcal{E}_{r+2,r+2}-\mathcal{E}_{r+3,r+3}\right)\,,\\ \Gamma(H_{\underline{3}}) &= \frac{1}{\sqrt{r(r-1)}}\left(\mathcal{E}_{r-1,r-1}+\mathcal{E}_{r,r}-\mathcal{E}_{r+1,r+1}-\mathcal{E}_{r+2,r+2}\right)\,,\\ \Gamma(H_{\underline{i}}) &= \frac{1}{\sqrt{r(r-1)}}\left(\mathcal{E}_{r+1-i,r+1-i}-\mathcal{E}_{r+2-i,r+2-i}+\mathcal{E}_{r-1+i,r-1+i}-\mathcal{E}_{r+i,r+i}\right)\,,\\ &\quad \forall i=4,\ldots,r\,.\end{aligned}$$

The matrices representing the ladder operators of $\mathfrak{d}_r$, and solving in particular $[\Gamma(H_{\underline{i}}),\Gamma(E_{\underline{j}})]=A_{\underline{ji}}\Gamma(E_{\underline{j}})$, can be determined to be (see [72])

$$\begin{aligned}\Gamma(E_{\varepsilon_i-\varepsilon_j}) &= \frac{1}{\sqrt{r(r-1)}}\left(\mathcal{E}_{i,j}+(-1)^{i+j+1}\mathcal{E}_{2r+1-j,2r+1-i}\right)\,,\\ \Gamma(E_{\varepsilon_i+\varepsilon_j}) &= \frac{1}{\sqrt{r(r-1)}}\left(\mathcal{E}_{i,2r+1-j}+(-1)^{i+j+1}\mathcal{E}_{j,2r+1-i}\right)\,.\end{aligned} \tag{C.4}$$

Raising and lowering operators in the basis $\{\beta_i\}_{i=1,..,r}$ then readily follow from relations (C.2) and (C.3) and expressions (C.4).

Finally, this representation of $\mathfrak{d}_r$ preserves the metric $G_{D_r}=\begin{pmatrix}\mathbb{O} & g_{D_r}^\top\\ g_{D_r} & \mathbb{O}\end{pmatrix}$, where the off-diagonal blocs are given by $g_{D_r}=$offdiag$\{1,-1,1,-1,\ldots,(-1)^{r-1}\}$. It can be checked that indeed: $\Gamma(X)^\top G_{D_r}+G_{D_r}\Gamma(X)=0$, for $X\in\mathfrak{d}_r$.

iii) Four involutive automorphisms for the real form $\mathfrak{so}(8,6)$: the set $\Delta_{(+1)}$ of roots generating the maximal compact subalgebra of the real form $\mathfrak{so}(8,6)$ appearing in Section 6.3 is determined for the four involutive automorphisms (6.31). Since $\dim\Delta_+(\mathfrak{d}_7)=42$, and since all four cases have $\dim\Delta_{(+1)}=18$, the corresponding involutive automorphisms all have signature $\sigma=5$, and thus determine isomorphic real forms, equivalent to $\mathfrak{so}(8,6)$. This construction lifts to the affine extension $\hat{\mathfrak{d}}_7$ through the automorphism (6.21) building the Cartan decomposition (6.33) and (6.34).

Herebelow, we give the set of roots $\Delta_{(+1)}$ for the four cases (6.31) explicitly. We remind the reader that these four involutive automorphisms all have $e^{\beta_2'(\overline{H})}=+1$ and $e^{\beta'_{i\neq 2,4,6}(\overline{H})}=-1$. Moreover, the set $\Delta_{(+1)}$ generating the non-compact vector space $\mathfrak{p}$ (6.34) can be deduced from $\Delta_{(-1)}=\Delta_+\backslash\Delta_{(+1)}$, where Δ_+ is obtained from the system (C.2) and (C.3) by setting $r=7$. In this case obviously $\dim\Delta_{(-1)}=24$.

The first involutive automorphism defined by $e^{\beta'_4(\overline{H})} = e^{\beta'_6(\overline{H})} = +1$ has

$$\begin{aligned}\Delta_{(+1)} = \big\{&\beta'_6, \beta'_4, \beta'_2, \beta'_{42}, \beta'_{765}, \beta'_{123}, \beta'_{7564}, \beta'_{5412}, \beta'_{5423}, \beta'_{4123}, \beta'_{76542}, \beta'_{65412}, \\ &\beta'_{65423}, \beta'_{412^23}, \beta'_{7654123}, \beta'_{765412^23}, \beta'_{7654^212^23}, \beta'_{65^24^212^23}\big\}\,. \end{aligned} \tag{C.5}$$

The second, defined by $e^{\beta'_4(\overline{H})} = -e^{\beta'_6(\overline{H})} = +1$ has

$$\begin{aligned}\Delta_{(+1)} = \big\{&\beta'_4, \beta'_2, \beta'_{76}, \beta'_{65}, \beta'_{42}, \beta'_{654}, \beta'_{123}, \beta'_{6542}, \beta'_{5412}, \beta'_{5423}, \beta'_{4123}, \beta'_{412^23}, \\ &\beta'_{765412}, \beta'_{765423}, b'_{654123}, \beta'_{65412^23}, \beta'_{654^212^23}, \beta'_{765^24^212^23}\big\}\,. \end{aligned} \tag{C.6}$$

The third, defined by $e^{\beta'_4(\overline{H})} = -e^{\beta'_6(\overline{H})} = -1$ has

$$\begin{aligned}\Delta_{(+1)} = \big\{&\beta'_6, \beta'_2, \beta'_{54}, \beta'_{765}, \beta'_{654}, \beta'_{542}, \beta'_{412}, \beta'_{423}, \beta'_{123}, \beta'_{6542}, \beta'_{54123}, \beta'_{765412}, \\ &\beta'_{765423}, \beta'_{654123}, \beta'_{5412^23}, \beta'_{65412^23}, \beta'_{7654^212^23}, \beta'_{65^24^212^23}\big\}\,. \end{aligned} \tag{C.7}$$

The fourth, defined by $e^{\beta'_4(\overline{H})} = e^{\beta'_6(\overline{H})} = -1$ has

$$\begin{aligned}\Delta_{(+1)} = \big\{&\beta'_2, \beta'_{76}, \beta'_{65}, \beta'_{54}, \beta'_{542}, \beta'_{412}, \beta'_{423}, \beta'_{123}, \beta'_{7654}, \beta'_{76542}, \beta'_{65412}, \beta'_{65423}, \\ &\beta'_{54123}, \beta'_{5412^23}, \beta'_{7654123}, \beta'_{765412^23}, \beta'_{654^212^23}, \beta'_{765^24^212^23}\big\}\,. \end{aligned} \tag{C.8}$$

The four of them lead as expected to $\dim \Delta_{(+1)} = 18$.

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q=2

n	Orbifold	shift vector	Q_1	$\lvert\xi\rvert^2$	n	Orbifold	shift vector	Q_1	$\lvert\xi\rvert^2$
3	$T^2/\mathbb{Z}_3$	$(3,3,3,3,3,3,3,3,2,1)$	1	-4	5	$T^2/\mathbb{Z}_5$	$(5,5,5,5,5,5,5,5,4,1)$	1	-8
						$T^2/\mathbb{Z}_5'$	$(5,5,5,5,5,5,5,5,3,2)$	2	-12
4	$T^2/\mathbb{Z}_4$	$(4,4,4,4,4,4,4,4,3,1)$	1	-6	6	$T^2/\mathbb{Z}_6$	$(6,6,6,6,6,6,6,6,5,1)$	1	-10
	$T^2/\mathbb{Z}_2'$	$(4,4,4,4,4,4,4,4,2,2)$	2	-8		$T^2/\mathbb{Z}_3'$	$(6,6,6,6,6,6,6,6,4,2)$	2	-16
						$T^2/\mathbb{Z}_2''$	$(6,6,6,6,6,6,6,6,3,3)$	3	-18

q=4

n	Orbifold	shift vector	(Q_1,Q_2)	$\lvert\xi\rvert^2$
3	$T^4/\mathbb{Z}_3$	$(3,3,3,3,3,3,4,2,2,1)$	$(1,-1)$	-2
4	$T^4/\mathbb{Z}_4$	$(4,4,4,4,4,4,5,3,3,1)$	$(1,-1)$	-4
	$T^2/\mathbb{Z}_4\times T^2/\mathbb{Z}_2$	$(4,4,4,4,4,4,5,3,2,2)$	$(1,-2)$	-6
	$T^4/\mathbb{Z}_2$	$(4,4,4,4,4,4,6,2,2,2)$	$(2,-2)$	0
	$T^2/\mathbb{Z}_2\times T^2/\mathbb{Z}_4$	$(4,4,4,4,4,4,6,2,3,1)$	$(2,-1)$	2
5	$T^4/\mathbb{Z}_5$	$(5,5,5,5,5,5,6,4,4,1)$	$(1,-1)$	-6
	$T^4/\mathbb{Z}_5'$	$(5,5,5,5,5,5,6,4,3,2)$	$(1,-2)$	-10
	$T^4/\mathbb{Z}_5''$	$(5,5,5,5,5,5,7,3,4,1)$	$(2,-1)$	0
	$T^4/\mathbb{Z}_5'''$	$(5,5,5,5,5,5,7,3,3,2)$	$(2,-2)$	-2
6	$T^4/\mathbb{Z}_6$	$(6,6,6,6,6,6,7,5,5,1)$	$(1,-1)$	-8
	$T^2/\mathbb{Z}_3\times T^2/\mathbb{Z}_6$	$(6,6,6,6,6,6,8,4,5,1)$	$(2,-1)$	2
	$T^2/\mathbb{Z}_6\times T^2/\mathbb{Z}_3$	$(6,6,6,6,6,6,7,5,4,2)$	$(1,-2)$	-14
	$T^4/\mathbb{Z}_3'$	$(6,6,6,6,6,6,8,4,4,2)$	$(2,-2)$	-8
	$T^2/\mathbb{Z}_2\times T^2/\mathbb{Z}_3$	$(6,6,6,6,6,6,9,3,4,2)$	$(3,-2)$	2
	$T^2/\mathbb{Z}_6\times T^2/\mathbb{Z}_2$	$(6,6,6,6,6,6,7,5,3,3)$	$(1,-3)$	-16
	$T^4/\mathbb{Z}_2'$	$(6,6,6,6,6,6,9,3,3,3)$	$(3,-3)$	0

q=6

n	Orbifold	shift vector	(Q_1,Q_2,Q_3)	$\lvert\xi\rvert^2$
3	$T^6/\mathbb{Z}_3$	$(3,3,3,3,2,4,4,2,2,1)$	$(-1,1,-1)$	0
4	$T^6/\mathbb{Z}_4$	$(4,4,4,4,3,5,5,3,3,1)$	$(-1,1,-1)$	-4
	$T^4/\mathbb{Z}_4\times T^2/\mathbb{Z}_2$	$(4,4,4,4,3,5,5,3,2,2)$	$(1,1,-2)$	-4
	$T^2/\mathbb{Z}_4\times T^4/\mathbb{Z}_2$	$(4,4,4,4,5,3,6,2,2,2)$	$(1,-2,2)$	2
5	$T^6/\mathbb{Z}_5$	$(5,5,5,5,4,6,6,4,4,1)$	$(-1,1,-1)$	-4
	$T^6/\mathbb{Z}_5'$	$(5,5,5,5,4,6,6,4,3,2)$	$(-1,1,-2)$	-8
	$T^6/\mathbb{Z}_5''$	$(5,5,5,5,4,6,7,3,4,1)$	$(-1,2,-1)$	2
	$T^6/\mathbb{Z}_5'''$	$(5,5,5,5,4,6,7,3,3,2)$	$(-1,2,-2)$	-2
6	$T^6/\mathbb{Z}_6$	$(6,6,6,6,5,7,7,5,5,1)$	$(-1,1,-1)$	-6
	$T^2/\mathbb{Z}_3\times T^4/\mathbb{Z}_6$	$(6,6,6,6,4,8,7,5,5,1)$	$(-2,1,-1)$	2
	$T^4/\mathbb{Z}_6\times T^2/\mathbb{Z}_3$	$(6,6,6,6,5,7,7,5,4,2)$	$(-1,1,-2)$	-12
	$T^2/\mathbb{Z}_6\times T^4/\mathbb{Z}_3$	$(6,6,6,6,5,7,8,4,4,2)$	$(-1,2,-2)$	-6
	$T^6/\mathbb{Z}_3'$	$(6,6,6,6,4,8,8,4,4,2)$	$(-2,2,-2)$	0
	$T^4/\mathbb{Z}_6\times T^2/\mathbb{Z}_2$	$(6,6,6,6,5,7,7,5,3,3)$	$(-1,1,-3)$	-14
	$T^2/\mathbb{Z}_6\times T^4/\mathbb{Z}_2$	$(6,6,6,6,5,7,9,3,3,3)$	$(-1,3,-3)$	2

Table 14: $\mathfrak{e}_{10}$ roots as class representatives of shift vectors for $\mathbb{Z}_n$ orbifolds.

q=8

n	Orbifold	shift vector	$(Q_1,\ldots,Q_4)$	$\lvert\xi\rvert^2$
3	$T^8/\mathbb{Z}_3$	$(3,3,2,4,2,4,4,2,2,1)$	$(1,-1,1,-1)$	2
4	$T^8/\mathbb{Z}_4$	$(4,4,5,3,3,5,5,3,3,1)$	$(1,-1,1,-1)$	-2
	$T^6/\mathbb{Z}_4\times T^2/\mathbb{Z}_2$	$(4,4,5,3,3,5,5,3,2,2)$	$(1,-1,1,-2)$	-2
5	$T^8/\mathbb{Z}_5$	$(5,5,6,4,4,6,6,4,4,1)$	$(1,-1,1,-1)$	-2
	$T^8/\mathbb{Z}_5'$	$(5,5,6,4,4,6,6,4,3,2)$	$(1,-1,1,-2)$	-6
	$T^8/\mathbb{Z}_5'''$	$(5,5,6,4,4,6,7,3,3,2)$	$(1,-1,2,-2)$	0
6	$T^8/\mathbb{Z}_6$	$(6,6,7,5,5,7,7,5,5,1)$	$(1,-1,1,-1)$	-4
	$T^2/\mathbb{Z}_3\times T^6/\mathbb{Z}_6$	$(6,6,8,4,5,7,7,5,5,1)$	$(2,-1,1,-1)$	2
	$T^6/\mathbb{Z}_6\times T^2/\mathbb{Z}_3$	$(6,6,7,5,5,7,7,5,4,2)$	$(1,-1,1,-2)$	-10
	$T^4/\mathbb{Z}_6\times T^4/\mathbb{Z}_3$	$(6,6,7,5,5,7,8,4,4,2)$	$(1,-1,2,-2)$	-4
	$T^2/\mathbb{Z}_6\times T^6/\mathbb{Z}_3$	$(6,6,7,5,4,8,8,4,4,2)$	$(1,-2,2,-2)$	2
	$T^6/\mathbb{Z}_6\times T^2/\mathbb{Z}_2$	$(6,6,7,5,5,7,7,5,3,3)$	$(1,-1,1,-3)$	-12
	$T^4/\mathbb{Z}_6\times T^2/\mathbb{Z}_3\times T^2/\mathbb{Z}_2$	$(6,6,7,5,5,7,8,4,3,3)$	$(1,-1,2,-3)$	-6
	$T^2/\mathbb{Z}_6\times T^4/\mathbb{Z}_3\times T^2/\mathbb{Z}_2$	$(6,6,7,5,4,8,8,4,3,3)$	$(1,-2,2,-3)$	0

q=10

n	Orbifold	shift vector	$(Q_1,\ldots,Q_5)$	$\lvert\xi\rvert^2$
4	$T^{10}/\mathbb{Z}_4$	$(3,5,5,3,3,5,5,3,3,1)$	$(-1,1,-1,1,-1)$	2
	$T^8/\mathbb{Z}_4\times T^2/\mathbb{Z}_2$	$(3,5,5,3,3,5,5,3,2,2)$	$(-1,1,-1,1,-2)$	0
5	$T^{10}/\mathbb{Z}_5$	$(4,6,6,4,4,6,6,4,4,1)$	$(-1,1,-1,1,-1)$	0
5	$T^{10}/\mathbb{Z}_5$	$(4,6,6,4,4,6,6,4,3,2)$	$(-1,1,-1,1,-2)$	-4
	$T^{10}/\mathbb{Z}_5'''$	$(4,6,6,4,4,6,7,3,3,2)$	$(-1,1,-1,2,-2)$	2
6	$T^{10}/\mathbb{Z}_6$	$(5,7,7,5,5,7,7,5,5,1)$	$(-1,1,-1,1,-1)$	-2
	$T^8/\mathbb{Z}_6\times T^2/\mathbb{Z}_3$	$(5,7,7,5,5,7,7,5,4,2)$	$(-1,1,-1,1,-2)$	-8
	$T^6/\mathbb{Z}_6\times T^4/\mathbb{Z}_3$	$(5,7,7,5,5,7,8,4,4,2)$	$(-1,1,-1,2,-2)$	-2
	$T^8/\mathbb{Z}_6\times T^2/\mathbb{Z}_2$	$(5,7,7,5,5,7,7,5,3,3)$	$(-1,1,-1,1,-3)$	-10
	$T^6/\mathbb{Z}_6\times T^2/\mathbb{Z}_3\times T^2/\mathbb{Z}_2$	$(5,7,7,5,5,7,8,4,3,3)$	$(-1,1,-1,2,-3)$	-4
	$T^4/\mathbb{Z}_6\times T^4/\mathbb{Z}_3\times T^2/\mathbb{Z}_2$	$(5,7,7,5,4,8,8,4,3,3)$	$(-1,1,-2,2,-3)$	2

Table 15: $\mathfrak{e}_{10}$ roots as class representatives of shift vectors for $\mathbb{Z}_n$ orbifolds.

q	$\xi^{[q,2]}$	physical basis	Dynkin labels
1	$\alpha_{(-1)^2 0^4 1^6 2^8 3^{10} 4^{12} 5^{14} 6^9 7^6 8^7}$	$(2,2,2,2,2,2,2,2,4,1)$	[200000001]
4	$\alpha_{(-1)^2 0^4 1^6 2^8 3^{10} 4^{12} 5^{13} 6^8 7^5 8^6}$	$(2,2,2,2,2,2,1,1,3,1)$	[100000100]
5	$\alpha_{(-1)^2 0^4 1^6 2^8 3^{10} 4^{11} 5^{12} 6^8 7^4 8^5}$	$(2,2,2,2,2,1,1,1,1,1)$	[000010000]
8	$\alpha_{(-1)^2 0^4 1^5 2^6 3^7 4^8 5^9 6^6 7^3 8^4}$	$(2,2,1,1,1,1,1,1,1,1)$	[010000000]
9	δ	$(0,1,1,1,1,1,1,1,1,1)$	[000000001]

Table 16: Physical class representatives for $T^{10-q} \times T^q/\mathbb{Z}_2$ orbifolds of M-theory of the first kind

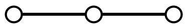
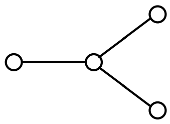
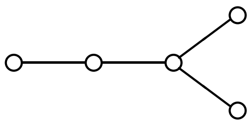
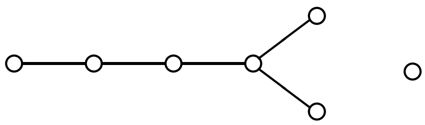
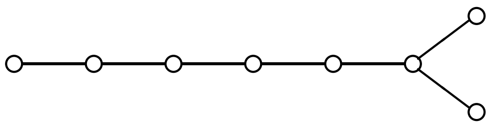
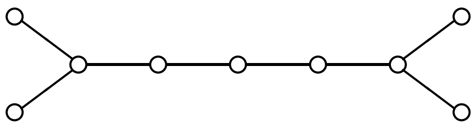
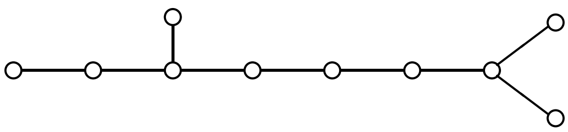
D	(Π_0, ϕ)	$\mathfrak{g}_{\rm inv}$	$\sigma(\mathfrak{g}_{\rm inv})$
8	 $\times$ H	$\mathfrak{so}(2,2) \oplus \mathfrak{so}(1,1)$	3
7	 $\times$ H	$\mathfrak{so}(3,3) \oplus \mathfrak{so}(1,1)$	4
6	 $\times$ H	$\mathfrak{so}(4,4) \oplus \mathfrak{so}(1,1)$	5
5	 $\times$ H	$\mathfrak{so}(5,5) \oplus \mathfrak{so}(1,1)$	6
4		$\mathfrak{so}(6,6) \oplus \mathfrak{sl}(2,\mathbb{R})$	7
3		$\mathfrak{so}(8,8)$	8
2		$\hat{\mathfrak{d}}_{8\|10}$	10
1		$\mathfrak{de}_{10\|10}$	10

Table 17: The split subalgebras $\mathfrak{g}_{\rm inv}$ for $\mathbb{Z}_2$ orbifolds of the first kind.

q	$\tilde{\xi}^{[q,2]}$	physical basis	generator
2	α_7	$(0,0,0,0,0,0,0,0,1,-1)$	$\mathcal{K}_{[9\,10]}$
3	α_8	$(0,0,0,0,0,0,0,1,1,1)$	$\mathcal{Z}_{[89\,10]}$
6	$\alpha_{34^25^36^278^2}$	$(0,0,0,0,1,1,1,1,1,1)$	$\widetilde{\mathcal{Z}}_{[5\cdots10]}$
7	$\alpha_{1^22^33^44^55^66^47^28^3}$	$(0,0,2,1,1,1,1,1,1,1)$	$\widetilde{\mathcal{K}}_{(3)[3\cdots10]}$

Table 18: Universal shift vectors for $\mathbb{Z}_2$ orbifolds of the second kind.

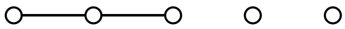
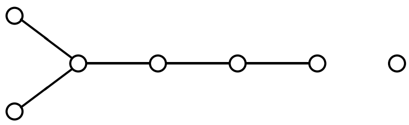
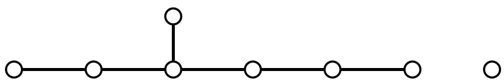
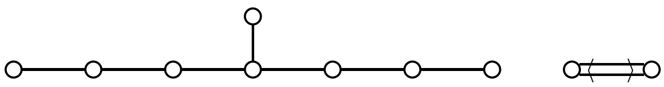
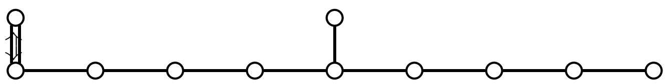
D	(Π_0, ϕ)	$\mathfrak{g}_{\text{inv}}$	$\sigma(\mathfrak{g}_{\text{inv}})$
8		$\mathfrak{sl}(2,\mathbb{R})^{\oplus 2} \oplus \mathfrak{so}(1,1)$	3
7		$\mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ $\oplus\, \mathfrak{so}(1,1)$	4
6		$\mathfrak{sl}(4,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})^{\oplus 2}$	5
5		$\mathfrak{sl}(6,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$	6
4		$\mathfrak{so}(6,6) \oplus \mathfrak{sl}(2,\mathbb{R})$	7
3		$\mathfrak{e}_{7\|7} \oplus \mathfrak{sl}(2,\mathbb{R})$	8
2		$\hat{\mathfrak{e}}_{7\|9} \oplus \hat{\mathfrak{a}}_{1\|3}$	10
1		${}^2\mathcal{KM}_{11\|12}$	10

Table 19: The split subalgebras $\mathfrak{g}_{\text{inv}}$ for $\mathbb{Z}_2$ orbifolds of the second kind

q	$\xi^{[q,2]}$	physical basis	Dynkin label	$\|\Lambda\|^2$
2	$\alpha_{(-1)^2 0^4 1^6 2^8 3^{10} 4^{12} 5^{14} 6^{10} 7^5 8^6}$	$(2, 2, 2, 2, 2, 2, 2, 2, 1, 1)$	[000000010]	-2
3	$\alpha_{(-1)^2 0^4 1^6 2^8 3^{10} 4^{12} 5^{15} 6^{11} 7^5 8^7}$	$(2, 2, 2, 2, 2, 2, 3, 3, 1, 2)$	[010000001]	-2
6	$\alpha_{(-1)^2 0^4 1^6 2^8 3^9 4^{12} 5^{15} 6^{10} 7^5 8^6}$	$(2, 2, 2, 2, 1, 3, 3, 1, 1, 1)$	[010001000]	2
7	$\alpha_{(-1)^2 0^4 1^7 2^{10} 3^{11} 4^{14} 5^{17} 6^{11} 7^5 8^7}$	$(2, 2, 3, 3, 1, 3, 3, 1, 1, 2)$	[000100100]	2

Table 20: Physical class representatives for $T^{10-q} \times T^q/\mathbb{Z}_2$ orbifolds of M-theory of the second kind